

Some Topological Properties of Star Connected Cycles

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Abstract

Star connected cycles are shown to be an undirected Cayley graph, and the graph automorphism group is determined. A routing algorithm is given, which finds an optimal path in polynomial time. The diameter, and tight bounds on the average distance, are computed.

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1 Introduction

The cube connected cycles graph was introduced in [12] as an interconnection network of interest. Since then various authors have studied it [9–11], and other “hypercube derivative” networks. The star graph was introduced in [1], and has since been studied by several authors [4,7]. In [8], star connected cycles, a derivative of the star graph, were introduced. Here we continue the study of these. They are shown to be a Cayley graph, and the automorphism group is determined. A routing algorithm is given, which finds an optimal path in polynomial time (this is done for the cube connected cycles in [10]). The diameter, and tight bounds on the average distance, are computed.

2 Star graphs

A Cayley graph is a directed graph whose vertices are the elements of a group G . The edges are determined by a set $\Gamma \subseteq G$; namely the edges are $\langle v, vg \rangle$ for $v \in G, g \in \Gamma$. The edge $\langle v, vg \rangle$ may be labelled with g . If Γ generates G then the digraph is strongly connected. If Γ is closed under inverses the digraph is symmetric, and may be considered undirected. If the elements of Γ have order 2 the labelled digraph may be considered undirected.

The star graph is a Cayley graph whose underlying group is the permutation group on $\{1, \dots, n\}$, which we denote S_n . The conventions for S_n which we will use are as follows. The elements are the bijections from $\{1, \dots, n\}$ to $\{1, \dots, n\}$. The product pq of two elements p and q is their composition $p \circ q$. If i_0, \dots, i_{s-1} are distinct elements of $\{1, \dots, n\}$ the notation $(i_0 \cdots i_{s-1})$ denotes the permutation p where $p[i_j] = i_{j+1}$ and $j+1$ is taken mod s ; and $p[t] = t$ for t not one of the i_j . Such a permutation is called a cycle, of length s ; a cycle of length 2 is called a transposition.

An element i is said to be fixed or moved by a permutation p , according to whether $p[i] = i$ or $p[i] \neq i$. Calling the domain of a cycle the elements it moves, any permutation may be written in an essentially unique way as a product of cycles with disjoint domains; this is well known and easy to see. In writing p in this way, fixed points may be omitted rather than written as cycles of length 1. We use λ to denote the identity permutation (the empty product of cycles).

A permutation $p \in S_n$ may also be denoted as the "string" $p_1 \cdots p_n$ where $p_i = p[i]$. For example, the cycle notation for 2341 is $(1\ 2\ 3\ 4)$. A convenient fact is that, in the string for $p \circ (i\ j)$ the elements in places i and j are transposed from the string for p ; and in $(i\ j) \circ p$ the elements i and j are transposed. (Note that omitting the composition sign causes no confusion as long as the cycles have length greater than 1.)

The permutations $(1\ i), 2 \leq i \leq n$, are easily seen to generate S_n (any permutation is a product of transpositions and $(i\ j) = (1\ i) \circ (1\ j) \circ (1\ i)$). The star graph, which we also denote as S_n , is the Cayley graph determined by these generators. It is so named because the graph whose vertices are $\{1, \dots, n\}$, and whose edges are the pairs of the transpositions, is a star. Note that p is adjacent to those vertices $p \circ (1\ i)$ obtained by exchanging (in the string for p) the element in the first place with the element in the i th place, $2 \leq i \leq n$.

Given $p \in S_n$ let c denote the number of nontrivial cycles in the cycle decomposition, and m the sum of their lengths; and let $f = 1$ if $p_1 \neq 1$ else 0. The length of a shortest path from p to λ is $c + m - 2f$. Further, all such paths are determined by the executions of the following nondeterministic algorithm.

Repeat until $p = \lambda$:

 If $p_1 = 1$ choose any i with $p_i \neq i$ and set $p = p \circ (1\ i)$;

 else

 either set $p = p \circ (1\ p_1)$,

 or choose any i in some nontrivial cycle not moving 1 and set $p = p \circ (1\ i)$.

These facts are noted in [1], and are easily proved by observing that right multiplication by any $(1\ i)$ can change $c + m - 2f$ by at most -1 ; the algorithm allows exactly the ways of achieving -1 ; and $m + c - 2f = 0$ iff $p = \lambda$.

3 Star connected cycles

The star connected cycles graph SCC_n is defined for $n \geq 2$, and is obtained from the star graph S_n by replacing each node by an $(n - 1)$ -vertex cycle. To define it, a variation of the notation is convenient, which we will adopt for the rest of the paper. S_n is considered to act on $Z_{n-1} \cup \infty$ where Z_{n-1} is the cyclic group of order $n - 1$. The generators for the graph S_n are $(i\ \infty)$ for $i \in Z_{n-1}$. Thus, $\{1, 2, \dots, n\}$ is replaced by $\{\infty, 0, \dots, n - 2\}$. One reason the former convention is frequently used is that S_{n-1} is canonically embedded as a subset of S_n , as the permutations fixing n . For SCC_n , the latter notation has many advantages.

The nodes of SCC_n are the pairs $\langle p, i \rangle$ where $p \in S_n$ and $i \in Z_{n-1}$. There are edges from $\langle p, i \rangle$ to $\langle p, i \pm 1 \rangle$ (called intra-cycle edges), and from $\langle p, i \rangle$ to $\langle p \circ (i\ \infty), i \rangle$ (called inter-cycle edges). The subgraph induced on $\{\langle p, i \rangle\}$ for fixed p is a cycle (called the p -cycle). The relation of belonging to the same p -cycle yields S_n as a quotient of SCC_n . In particular, a path in SCC_n induces a path in S_n , namely visiting the p of the $\langle p, i \rangle$.

SCC_n is analogous to the cube connected cycles HCC_n , where each node of the hypercube H_n is replaced by an n -vertex cycle. It is well known that HCC_n is a Cayley graph [6]; an analogous argument shows that SCC_n is also a Cayley graph. For the following let ρ be the permutation where $\rho[i] = i + 1$ for $i \in Z_{n-1}$ and $\rho[\infty] = \infty$.

Theorem 1 SCC_n is a Cayley graph.

Proof. The group is a semi-direct product (q.v. see [5]) of S_n by Z_{n-1} . The map from Z_{n-1} to the automorphism group of S_n is that which takes i to conjugation by ρ^i . To write the edges as $x \rightarrow xe$, we write the product as follows.

$$\langle p, i \rangle \langle q, j \rangle = \langle p \circ \rho^i \circ q \circ \rho^{-i}, i + j \rangle.$$

The generators are $\langle \lambda, \pm 1 \rangle$ and $\langle (0 \ \infty), 0 \rangle$. \square

As is well known, if G is a Cayley graph and $u \in G$ then the map $x \mapsto ux$, left translation by u , is an automorphism of the graph; these form a group, which acts transitively on the vertices of G . There is another automorphism of SCC_n , which fixes $\langle \lambda, 0 \rangle$ and acts on the generators by exchanging $\langle \lambda, \pm 1 \rangle$. Let τ be the permutation where $\tau[i] = -i$ for $i \in Z_{n-1}$ and $\tau[\infty] = \infty$. Note that ρ and τ , considered as acting on Z_{n-1} , generate the dihedral group; in particular τ has order 2 and $\tau \circ \rho^j = \rho^{-j} \circ \tau$. Let A_τ be the map $\langle p, i \rangle \mapsto \langle \tau \circ p \circ \tau, -i \rangle$ on SCC_n . It is readily verified that A_τ is a group automorphism of order 2 with the properties stated above, whence a graph automorphism.

Theorem 2 *The automorphism group of the graph SCC_n equals the product of the subgroup of left translations, and the subgroup of order 2 generated by A_τ . The edges fall into two orbits, the intra-cycle edges and the inter-cycle edges.*

Proof. Writing A_u for $x \mapsto ux$, it is readily verified that

$$A_{\langle q, j \rangle} A_\tau = A_\tau A_{\langle \tau \circ q \circ \tau, -j \rangle},$$

whence the two subgroups commute. Each subgroup acts on the two types of edges, and it is readily verified that the action of their product on each type is transitive. To complete the proof it suffices to show that if a graph automorphism fixes $\langle \lambda, 0 \rangle$ and the edges incident to it then it fixes every vertex. Suppose $n \geq 3$ and $n \neq 13$. By considering the path in S_n , it is easy to check that an intra-cycle edge lies on a unique cycle of length 12 of SCC_n , with the cycles being opposite for opposite edges. From this it follows that if $\langle p, i - 1 \rangle$ and $\langle p, i \rangle$ are fixed then (since $\langle p \circ (i \ \infty), i \rangle$ is fixed) $\langle p, i + 1 \rangle$ is fixed. It follows that if two adjacent vertices of a p -cycle are fixed, the entire p -cycle is fixed. Moreover, if the p -cycle is fixed then the $p \circ (i \ \infty)$ -cycle is also fixed, and the theorem follows. If $n = 2$ the theorem is clear. If $n = 13$, it is clear that both cycles of length 12 containing $\langle \lambda, 0 \rangle$ and $\langle \lambda, 1 \rangle$ must be fixed. Generalizing this argument, the theorem follows in this case also. \square

More generally, each node of $H_m(S_m)$ where $m \leq n$ may be replaced by an $n(n-1)$ vertex cycle. These graphs have been considered in the hypercube case, and in the star case are used in [3]. We will not discuss them here, except to mention that they are induced subgraphs of $HCC_n(SCC_n)$, and do not in general have an automorphism group transitive on the vertices.

4 Distance in SCC_n

Let $\alpha(i)$ for $i \in Z_{n-1}$ equal i if $0 \leq i \leq (n-1)/2$, or $(n-1) - i$ otherwise. That is, $\alpha(i)$ is the distance around the cycle from 0. Note that $\alpha(-i) = \alpha(i)$. For $i, j \in Z_{n-1}$ let $\beta(i, j) = \alpha(i - j)$. This function satisfies the triangle inequality, as is readily verified; indeed it is a metric on Z_{n-1} . We allow a path in a graph to have repeated vertices in general, and call it simple if it does not.

Lemma 3 *If P is a minimum distance path in SCC_n then the induced path in S_n is simple.*

Proof. It suffices to observe that a path in SCC_n from $\langle p, i \rangle$ to $\langle p, j \rangle$ must have at least $\beta(i, j)$ intra-cycle edges. \square

Fix a vertex $\langle p, e \rangle$ of SCC_n . Given a simple path P in S_n from p to λ , let d_P be the minimum length of a simple path in SCC_n from $\langle p, e \rangle$ to $\langle \lambda, 0 \rangle$, whose induced path is P . It is easy to determine d_P . If $i_1 \cdots i_m$, $m > 0$, are the elements of Z_{n-1} such that $(i_l \infty)$ are the edge labels along P , then

$$d_P = m + \beta(e, i_1) + \beta(i_m, 0) + \sum_{l=1}^{m-1} \beta(i_l, i_{l+1}).$$

If $m = 0$ then $d_P = \beta(e, 0)$. Write d for the distance in SCC_n from $\langle p, e \rangle$ to $\langle \lambda, 0 \rangle$. By Lemma 3 $d = \min_P d_P$, where P ranges over simple paths in S_n from p to λ .

We will write $(\infty i_1 \cdots i_r)$ for a cycle moving ∞ , and $(i_1 \cdots i_r)$ for a cycle fixing ∞ . In the latter case, there are r choices for i_1 . Define $\chi_1(C)$ to be r or $r + 1$ as C moves or fixes ∞ , and let $\chi_1(p) = \sum_C \chi_1(C)$ where the sum is over the cycles of p . Clearly $\chi_1(p)$ equals the distance in S_n from p to λ . Define

$$\chi_2(C) = \begin{cases} \sum_{l=1}^{r-1} \beta(i_l, i_{l+1}) & \text{if } C \text{ moves } \infty, \\ \beta(i_r, i_1) + \sum_{l=1}^{r-1} \beta(i_l, i_{l+1}) & \text{if } C \text{ fixes } \infty. \end{cases}$$

Clearly, when C fixes ∞ , $\chi_2(C)$ does not depend on the choice of i_1 . Let $\chi_2(p) = \sum_C \chi_2(C)$, and let $\chi(p) = \chi_1(p) + \chi_2(p)$.

Write χ for $\chi(p)$; and if p moves ∞ let $\delta_1 = \beta(e, i_1) + \beta(i_r, 0)$, else let $\delta_1 = \beta(e, 0)$.

Theorem 4 *$d = \chi + \delta_1 + \delta_2$ where $0 \leq \delta_2 \leq 2\lfloor(n - 2 - \beta(e, 0))/2\rfloor$. A path minimizing d can be found in polynomial time.*

Proof. If p moves ∞ start with a sequence

$$e \rightarrow i_1 \rightarrow \cdots \rightarrow i_r \rightarrow 0;$$

otherwise start with a sequence $e \rightarrow 0$. A minimum length path P in S_n results in a sequence of replacements to the sequence, of the form, replace some pair $a \rightarrow b$ by the sequence

$$a \rightarrow i_r \rightarrow i_1 \rightarrow \cdots \rightarrow i_r \rightarrow b.$$

By the triangle inequality the sum over the sequence of the $\beta(a, b)$ for the pairs $a \rightarrow b$ is nondecreasing. Letting d^* denote the minimum over minimum distance P , this shows that $d^* \geq \chi + \delta_1$. Suppose that n is even, so that the shortest arc is unique. Define the range of a cycle fixing 1 to be the union of the arcs for the pairs i_l, i_{l+1} , $1 \leq l < r$, and the arc for i_r, i_1 ; again this does not depend on i_1 . Define the initial range to be the union of the arcs for the pairs along the initial sequence. Now, if in a substitution i_r lies on a shortest arc from a to b in Z_{n-1} then the sum of the β does not increase. It is easy to see that, applying such operations in any order will result in merging all the cycles whose ranges can be reached from the initial range by a sequence of overlaps. Similarly the remaining ranges can be merged, resulting in a set of disjoint ranges. If there is only one range after merging, let $\delta_2 = 0$; otherwise let $\delta_2 = 2 \sum' g_t$ where g_t is the gap sizes and the sum omits a largest gap size. Clearly $d^* = \chi + \delta_1 + \delta_2$; the gaps must be traversed twice, so an additional δ_2 intra-cycle steps are necessary, and also sufficient. The upper bound on δ_2 follows since the initial range covers at least $\beta(e, 0)$ edges of Z_{n-1} , and the remaining ranges at least 1. It remains to show that $d = d^*$; but by the triangle inequality and considering the cases it follows that extraneous transpositions cannot decrease the intra-cycle travel below $\chi_2 + \delta_1 + \delta_2$. Merging of ranges can readily be accomplished as follows. Denote each range as an ordered pair $\langle a, b \rangle$, where the ranges consists of travelling clockwise from a to b . Now sort the ranges according to a . The modifications for odd n are as follows. Call an adjacent pair i, j in a cycle of p with $\beta(i, j) = (n - 1)/2$ a diameter. It is readily seen that if there are three or more diameters then δ_2 can be made 0. If there are 1 or 2 diameters then all the possibilities may be checked, to minimize δ_2 . \square

5 Maximum and average distance

The maximum and average values of χ_1 are given in [1], and are

$$\max = \lfloor 3(n - 1)/2 \rfloor, \quad \text{avg} = n + \frac{2}{n} - 4 + h_n,$$

where h_n is the n th Harmonic number $\sum_{k=1}^n 1/k$. If n is even the permutations at maximum distance from λ either move ∞ and have $n/2$ transpositions, or fix ∞ and have $n/2 - 2$ transpositions and a cycle of length 3. If n is odd the permutations of maximum distance fix ∞ and have $(n - 1)/2$ transpositions. Some of the discussion to be given here of the maximum and average distance in SCC_n can be found in [8] and [2] respectively.

Theorem 5 *Let M denote the maximum value of α , namely $n/2 - 1$ if n is even and $(n - 1)/2$ if n is odd. The maximum value of $\chi_2 + \delta_1$ equals nM . It is achieved by a permutation which also maximizes χ_1 .*

Proof. The first claim can be seen by considering the cases of fixing and moving ∞ . The vertices for the second claim are

$$\begin{cases} \langle (0 \ (n-1)/2) \cdots ((n-3)/2 \ (n-2)), (n-1)/2 \rangle & n \text{ odd} \\ \langle (\infty \ (n/2-1))(0 \ n/2) \cdots ((n/2-3) \ (n-2)), 0 \rangle & n \text{ even.} \quad \square \end{cases}$$

Theorem 6 *Let A denote the average value of $\alpha(i)$, namely $(n^2 - 2n)/(4(n - 1))$ if n is even or $(n^2 - 2n + 1)/(4(n - 1))$ if n is odd. Using overbar to denote average,*

$$\bar{\chi}_2 = \frac{(n-1)^2}{n}A, \quad \bar{\delta}_1 = \frac{2n+1}{n}A.$$

Proof. It is easily seen that $\chi_2(p) = \sum_i \beta(i, p[i])$, where the sum is over those i such that $i \neq \infty$, $p[i] \neq \infty$, $p[i] \neq i$. Hence the average value equals

$$\begin{aligned} \frac{1}{n!} \sum_p \sum_i \beta(i, p[i]) &= \frac{1}{n!} \sum_{i,j \in Z_{n-1}, i \neq j} \beta(i, j) |\{p : p[i] = j\}| \\ &= \frac{1}{n!} \sum_{i \neq j} \beta(i, j) (n-1)! = \frac{1}{n} \sum_{i \neq j} \beta(i, j) = \frac{n-1}{n} \sum_{k \neq 0} \alpha(k). \end{aligned}$$

For the second claim, $\bar{\delta}_1$ equals

$$\frac{1}{n}D_1 + \frac{n-1}{n} \frac{1}{n-1}D_2 + \frac{n-1}{n} \frac{n-2}{n-1}D_3$$

where D_1 is the average if ∞ is fixed, D_2 the average if ∞ is in a transposition, and D_3 the average otherwise. Remaining details are left to the reader. \square

Theorem 5 yields a lower bound on the maximum distance. The true maximum in fact equals this, as similar arguments show. Theorem 6 yields a lower bound

on the average distance. The true average is at most $n - 2$ greater than this, and clearly the correction is smaller than $n - 2$. Indeed, let $\lambda(p)$ be the amount of Z_{n-1} which is covered by the arcs of p , say clockwise for diameters. It is easy to see that $\delta_2 \leq \lambda$. We conjecture that $\bar{\lambda}$ goes to 0 exponentially in n .

6 Conclusion

Studying derivatives of the hypercube has contributed to the study of the hypercube, and indeed interconnection networks in general. The star connected cycles are an interesting derivative of the star graph. Here we have continued the study of its topological properties. In particular, we have noted that the star connected cycles is a Cayley graph, and determined its automorphisms. We have given a polynomial time algorithm for finding a minimum distance path between two points; and computed the diameter and tight bounds on the average distance.

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