Appears in *Proceedings of the IEEE 15th Annual International Phoenix Conference on Computers and Communications, Phoenix, AZ, March 27-29, 1996, pp. 115-122.*

Low Expansion Packings and Embeddings of Hypercubes into Star Graphs*

Marcelo Moraes de Azevedo†, *Shahram Latifi*‡, *and Nader Bagherzadeh*†

†Dept. of Electrical and Computer Engineering
University of California – Irvine, CA 92717-2625

‡Dept. of Electrical and Computer Engineering
University of Nevada – Las Vegas, NV 89154-4026

Abstract — *Let $G(k)$ and $H(n)$ be respectively a k-dimensional and an n-dimensional graph.* Packing *is a technique by which p_k many copies of each $G(k)$, $k_{min} \leq k \leq k_{max}$, are embedded into $H(n)$. Packings can use $H(n)$ efficiently by assigning independent tasks to the embedded copies of $G(k)$, and are a useful foundation from which node allocation and task migration strategies can be built. Copies of $G(k)$, packed into $H(n)$ with dilation d_{base}, can be combined to produce a* variable-dilation embedding *of $G(k+\ell)$ into $H(n)$. Such an embedding has dilation d_i along dimension i of $G(k+\ell)$, where $d_i = d_{base}$ for $i \leq k$, and $d_i > d_{base}$ for $k < i \leq k+\ell$. The* average dilation *of the embedding is $d_{avr} = \frac{1}{k+\ell}\sum_{i=1}^{k+\ell} d_i$, and can often be made close to d_{base}.*

This paper focuses on the problem of packing hypercubes $Q(k)$ into a star graph $S(n)$ with load 1 and dilation 3. Our results include: 1) fixed-sized packings of $Q(k)$ into $S(n)$, 2) multiple-sized packings of $Q(k)$ into $S(n)$, and 3) variable-dilation embeddings of $Q(k+\ell)$ into $S(n)$. Our variable-dilation embeddings and multiple-sized packings provide flexibility to support tasks with a wide range of node allocation requirements. The expansion of our multiple-sized packings is optimal (i.e., 1), which solves the problem of growing expansion found in single embeddings of hypercubes into star graphs. Moreover, for $n \leq 10$, the average dilation of our variable-dilation embeddings lies between 3 and 4.25.

1 Introduction

The star graph [1] was proposed as an attractive interconnection network for parallel processing, featuring smaller degree and diameter than a hypercube [2] of comparable size. However, the earlier introduction of hypercube networks, along with their interesting characteristics, has given to such networks considerable popularity. A number of hypercube-configured parallel computers was built in recent years [2], and many hypercube-compatible algorithms have been proposed [3]. Despite the fact that some parallel algorithms have also been specifically devised for the star graph (e.g., sorting [4], FFT [5]), we believe that the repertory of star graphs algorithms can be significantly increased via hypercube embeddings.

Research on embedding hypercubes into star graphs was initiated by Nigam, Sahni, and Krishnamurthy [6]. The interesting techniques introduced in [6] reveal a major challenge for embedding hypercubes into star graphs. Namely, topological differences between the two networks (e.g., degree and minimum cycle length) make it difficult to obtain an embedding that simultaneously achieves small *dilation* and *expansion* (see Sec. 2 for definitions).

In this paper, we present a hypercube embedding technique that reduces the trade-off between dilation and expansion significantly. Our technique is referred to as *packing*, and consists of embedding a disjoint union $U = \bigcup_{k=k_{min}}^{k_{max}} \bigcup_{j=0}^{p_k-1} Q_j(k)$ containing p_k many copies of each k-dimensional hypercube $Q(k)$, $k_{min} \leq k \leq k_{max}$, into an n-dimensional star graph $S(n)$. Our packings lie in two categories, namely *fixed-sized packings* (i.e., packings in which all of the embedded hypercubes have the same size, or $k_{min} = k_{max}$) and *multiple-sized packings* (i.e., packings in which the embedded hypercubes are of various sizes, or $k_{min} \neq k_{max}$). All of our packings have load 1, i.e. any node in $S(n)$ is image to at most one node of U.

Packings support multiple tasks, providing a means by which a star graph can be efficiently used to run hypercube algorithms. Moreover, they can also be used as a foundation for implementing node allocation and task migration strategies, making it possible to handle such issues as load balancing and fault tolerance (see [7] for more on this topic).

The main contribution of our packing techniques relates to expansion, which can be either: 1) a slow growing function of n, in the case of fixed-sized packings (an expansion between 1 and 2.46 is obtained for $n \leq 10$), or 2) optimal (i.e., 1), in the case of multiple-sized packings. These small expansion ratios need not sacrifice dilation (e.g., Sec. 4 presents packing techniques that produce dilation 3 for all embedded hypercubes).

We also consider in this paper an extension of our packing techniques, which we refer to as *variable-dilation embeddings.* Variable-dilation embeddings address the necessity of accommodating tasks requiring hypercubes that are larger than each of the copies made available through our basic packing techniques. Larger hypercubes $Q(k+\ell)$ are formed from 2^ℓ packed copies of lower-dimensional hypercubes $Q(k)$, which assures that small expansion is still achieved. While these $Q(k)$'s are embedded with a fixed dilation $d_{base} = 3$, larger dilation (e.g., 4, 6, and so on) is produced along higher dimension links of $Q(k+\ell)$. However, the *average dilation* of such an embedding (d_{avr}) is not much larger than d_{base} (e.g., d_{avr} ranges from 3 to 4.25 for $k+\ell \leq 20$ and $n \leq 10$).

This paper is organized as follows. Sec. 2 introduces basic definitions and the terminology used in the paper. Sec. 3 presents some background information. Sec. 4 presents our techniques for packing hypercubes into the star graph. Sec. 5 discusses variable-dilation embeddings. A comparison with related work is given in Sec. 6 and Sec. 7 concludes the paper.

*This research is supported in part by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq - Brazil), under the grant No. 200392/92-1.

Appears in *Proceedings of the IEEE 15th Annual International Phoenix Conference on Computers and Communications, Phoenix, AZ, March 27-29, 1996, pp. 115-122.*

2 Definitions and Terminology

Let $G(k)$ be a k-dimensional graph with hierarchical structure, such that $G(k+1)$ is obtained recursively from $c(k)$ many copies of $G(k)$. Several graphs belonging to the class of *Cayley graphs* have this recursive decomposition property, such as the hypercube and the star graph [1, 2]. The links connecting the $c(k)$ copies of $G(k)$ that exist within $G(k+1)$ are referred to as *dimension* $(k+1)$ *links.*

We denote the set of nodes and the set of links of $G(k)$ by $V(G(k))$ and $E(G(k))$, respectively. An *embedding* of $G(k)$ into $H(n)$, which we denote by $F : G(k) \mapsto H(n)$, is a mapping of $V(G(k))$ into $V(H(n))$ and of $E(G(k))$ into paths of $H(n)$. $G(k)$ and $H(n)$ are respectively referred to as the *guest* and the *host of F* [3]. The *node image of F* is $F(G(k)) = \{F(u) : u \in V(G(k))\}$. The *load of F* is the maximum number of nodes of $G(k)$ that are mapped to any single node of $H(n)$, and is denoted by $\lambda(F)$. The *dilation of F* is $d(F) = \max\{dist_H(F(u), F(v)) : (u,v) \in E(G(k))\}$, where $dist_H(a,b)$ is the distance in $H(n)$ between two vertices a and b of $H(n)$. The *expansion of F* is $X(F) = |V(H(n))|/|V(G(k))|$.

Let $U = \bigcup_{k=k_{min}}^{k_{max}} \bigcup_{j=0}^{p_k-1} G_j(k)$ denote a disjoint union of p_k many copies of each $G(k)$, with k ranging from k_{min} to k_{max}. For each k, we index $G_j(k)$ with $0 \le j < p_k$. The set of nodes in U is $V(U) = \{u \in V(G_j(k)) : k_{min} \le k \le k_{max}, 0 \le j < p_k\}$. Accordingly, the set of links in U is $E(U) = \{(u,v) \in E(G_j(k)) : k_{min} \le k \le k_{max}, 0 \le j < p_k\}$. A *packing* of U into $H(n)$, which we denote by $P : U \mapsto H(n)$, is a mapping of $V(U)$ into $V(H(n))$ and of $E(U)$ into paths of $H(n)$. P is a *fixed-sized packing* if $k_{min} = k_{max}$. Otherwise, P is a *multiple-sized packing.* The *node image of P* is $P(U) = \{P(u) : u \in V(U)\}$. The *load of P* is the maximum number of nodes in U that are mapped to any single node of $H(n)$, and is denoted by $\lambda(P)$. We denote the embedding of $G_j(k)$ into $H(n)$, in the context of P, by $P_{j,k}$. The *dilation of* $P_{j,k}$ is $d(P_{j,k}) = \max\{dist_H(P(u), P(v)) : (u,v) \in E(G_j(k))\}$. The *base dilation of P* is $d_{base}(P) = \min\{d(P_{j,k}) : k_{min} \le k \le k_{max}, 0 \le j < p_k\}$. P is referred to as a *template packing* if $d(P_{j,k}) = d_{base}(P)$, for $k_{min} \le k \le k_{max}$, $0 \le j < p_k$. The *expansion of P*, denoted by $X(P)$, is:

$$X(P) = \frac{|V(H(n))|}{|V(U)|} = \frac{|V(H(n))|}{\sum_{k=k_{min}}^{k_{max}} (p_k \cdot |V(G(k))|)} \quad (1)$$

Embeddings of guest graphs whose dimensionality exceeds k_{max} can often be built from a packing P, and are defined as follows. For some $k > 0$ and $\ell > 0$, let $c(k \to k+\ell) = \prod_{i=1}^{\ell} c(k+i-1)$ denote the number of $G(k)$'s needed to hierarchically compose one $G(k+\ell)$. We denote the disjoint union of $G(k)$'s that compose $G_0(k+\ell)$ by $U_{0,k+\ell} = \bigcup_{j=j_0}^{j_0+c(k\to k+\ell)-1} G_j(k)$, where $U_{0,k+\ell} \subseteq U$. A *variable-dilation embedding* of $G_0(k+\ell)$ into $H(n)$, which we denote by $W : G_0(k+\ell) \mapsto H(n)$, is a mapping of $V(G_0(k+\ell))$ into $V(H(n))$ and of $E(G_0(k+\ell))$ into paths of $H(n)$, constrained by a packing $P : U \mapsto H(n)$, such that $\forall u \in V(U_{0,k+\ell})$, $W(u) = P(u)$. Equivalently, W can be thought of as applying a transformation to P as follows. Let $U' = U - U_{0,k+\ell} + G_0(k+\ell)$ be the disjoint union produced from U, when $c(k \to k+\ell)$ many $G(k)$'s hierarchically compose $G_0(k+\ell)$. A packing $P' : U' \mapsto H(n)$ is produced when W is constructed within P. In general, one can apply a series of similar transformations to P, producing a packing $P^{general}$ that will hold several variable-dilation embeddings. Previously given definitions also apply to packings holding variable-dilation embeddings. In particular, note that $\lambda(P') = \lambda(P)$, $X(P') = X(P)$, and $W = P'_{0,k+\ell}$.

Some terms that more precisely characterize a variable-dilation embedding are defined as follows. The *dilation of W along the* i^{th} *dimension of* $G_0(k+\ell)$ is $d_i(W) = \max\{dist_H(P'(u), P'(v)) : (u,v) \in E_i(G_0(k+\ell))\}$, where $E_i(G_0(k+\ell))$ denotes the set of dimension i links of $G_0(k+\ell)$, $1 \le i \le k+\ell$. Hence, $d(W) = \max\{d_i(W) : 1 \le i \le k+\ell\}$. The *dilation vector of W* is $\overline{d(W)} = [d_1(W), d_2(W), \ldots, d_{k+\ell}(W)]$. The *average dilation of W* is:

$$d_{avr}(W) = \frac{\sum_{i=1}^{k+\ell} d_i(W)}{k+\ell} \quad (2)$$

A major advantage of variable-dilation embeddings, as opposed to conventional embedding methods, is that the dilation can be made significantly smaller *on the average.* Since many algorithms use a limited number of dimensions at any given step of their execution, a smaller communication slowdown is obtained.

3 Background

3.1 The hypercube

A k-dimensional hypercube graph $Q(k) = \{V(Q(k)), E(Q(k))\}$ contains 2^k nodes, which are labeled with binary strings of length k. A node $\phi = q_1q_2\ldots q_i\ldots q_k$ is connected to k distinct nodes, respectively labeled with strings $\phi_i = q_1q_2\ldots\overline{q_i}\ldots q_k$, $1 \le i \le k$, where $\overline{q_i}$ denotes the binary negation of bit q_i [2]. The link connecting ϕ and ϕ_i is a *dimension i link* of $Q(k)$.

3.2 The star graph

An n-dimensional star graph $S(n) = \{V(S(n)), E(S(n))\}$ contains $n!$ nodes which are labeled with the $n!$ possible permutations of n distinct symbols. In this paper, we use the integers $\{1, 2, \ldots, n\}$ to label the nodes of $S(n)$. A node $\pi = p_1p_2\ldots p_i\ldots p_n$ is connected to $(n-1)$ distinct nodes, respectively labeled with permutations $\pi_i = p_ip_2\ldots p_{i-1}p_1p_{i+1}\ldots p_n$, $2 \le i \le n$ (i.e. π_i's label is obtained by exchanging the first and the i^{th} symbol of π's label) [1]. The link connecting π and π_i is a *dimension i link* of $S(n)$.

3.3 Embedding of a mesh into $S(n)$

The packing and embedding techniques presented in this paper use a two-step mapping algorithm, in which hypercubes are initially packed into an $(n-1)$-dimensional mesh of size $2 \times 3 \times \ldots \times n$, which we denote by $M(n-1) = \{V(M(n-1)), E(M(n-1))\}$. $M(n-1)$ is then embedded

Appears in *Proceedings of the IEEE 15th Annual International Phoenix Conference on Computers and Communications, Phoenix, AZ, March 27-29, 1996, pp. 115-122.*

into $S(n)$ with load 1, dilation 3, and expansion 1 via Algorithm 1 below, which is inspired by a mapping algorithm proposed by Ranka et al. [8]. An interesting property of Alg. 1 is presented later in this paper (see Lemma 2).

Algorithm 1 *(Mapping $M(n-1)$ onto $S(n)$):*

```
mesh_to_star (int n, m[ ], p[ ])
{ int i, h, temp;
  for (i = 1; i <= n; i = i + 1) p[i] = i;
  for (i = 1; i < n; i = i + 1)
    for (h = 0; h < m[i]; h = h + 1) {
      temp = p[i - h + 1];
      p[i - h + 1] = p[i - h];
      p[i - h] = temp; } }
```

Let $s_i = i + 1$ denote the width of $M(n-1)$ along its i^{th} dimension. We label the nodes of $M(n-1)$ with an $(n-1)$-integer vector $m_1 m_2 \dots m_{n-1}$, where $0 \leq m_i \leq s_i - 1$. Alg. 1 maps $\omega = m_1 m_2 \dots m_{n-1}$, $\omega \in V(M(n-1))$, onto $\pi = p_1 p_2 \dots p_n$, $\pi \in V(S(n))$. The pseudocode represents node labels with vectors, such that $\omega = m[\,]$ and $\pi = p[\,]$.

4 Template Packings

4.1 Preliminaries

In this section, we discuss *template* packings of hypercubes into $S(n)$, which have load 1 and base dilation 3. We present both fixed-sized and multiple-sized packings, which respectively embed into $S(n)$: 1) a disjoint union $U = \bigcup_{j=0}^{p_k-1} Q_j(k)$, for some fixed $k \in [\lfloor n/2 \rfloor, n-1]$, and 2) a disjoint union $U = \bigcup_{k=\lfloor n/2 \rfloor}^{n-1} \bigcup_{j=0}^{p_k-1} Q_j(k)$. $Q(n-1)$ is the largest hypercube considered in this paper as far as template packings into $S(n)$ are concerned. In addition, because fixed-sized packings of $Q(\lfloor n/2 \rfloor)$ and multiple-sized packings produce expansion 1, we do not discuss the cases $k < \lfloor n/2 \rfloor$, which can be obtained by tearing larger hypercubes after they are packed into $S(n)$.

From the viewpoint of how hypercubes are packed into $M(n-1)$, we classify our packings as *symmetric* or *asymmetric*. Symmetric packings are those in which all dimension a links of $E(U)$ are mapped to dimension b links of $E(M(n-1))$, where $1 \leq a \leq k$ and $1 \leq b < n$. Accordingly, asymmetric packings are those in which two dimension a links (u, v) and (x, y) of $E(U)$ may be mapped to links of different dimensions b and c in $E(M(n-1))$, unless (u, v) and (x, y) belong to the same $Q_j(k) \in U$.

The dimension mapping rules that characterize a given packing technique are not preserved when $M(n-1)$ is ultimately embedded into $S(n)$ via Alg. 1. In both symmetric and asymmetric template packings, a dimension a link of $E(U)$ is mapped either to a dimension b link of $E(S(n))$, or to a path $b \rightarrow c \rightarrow b$ in $S(n)$, where b, c can be any of the dimensions of $S(n)$. Although the symmetry (or asymmetry) of a particular technique used to pack $Q(k)$ into $S(n)$ can not be distinguished unless for the intermediary step where the hypercubes are packed into $M(n-1)$, we use throughout the paper the terms *symmetric* (or *asymmetric*) *packings of $Q(k)$ into $S(n)$*.

Our discussion about fixed-sized packings considers both the symmetric and the asymmetric cases. However, our multiple-sized packings are all asymmetric. Symmetric packings provide a very regular arrangement of the copies of $Q(k)$ in $M(n-1)$, which is particularly useful for constructing variable-dilation embeddings. However, they do not achieve as small expansion as asymmetric packings do. In order to combine the desired features of low expansion and support to variable-dilation embeddings, we build our asymmetric packings as an extension of their symmetric counterparts. Hence, an asymmetric packing will often be the method of choice to pack hypercubes into the star graph.

Intuitively, one should expect smaller expansion when: 1) smaller hypercubes are packed into $S(n)$, and 2) asymmetric techniques are used.

4.2 Embedding of $Q(k)$ into $M(n-1)$

Our packing techniques take advantage of the regular structure of $M(n-1)$ to achieve low expansion ratios. A preliminary result on which our techniques are based is given below:

Lemma 1 *$Q(k)$ can be embedded into $M(n-1)$ with load 1, dilation 1, if $k \leq n-1$.*

Proof: To prove the lemma, we present an algorithm which produces the desired embedding (see Alg. 2). Assume that the argument *origin*[ ] taken by Alg. 2 is set to all 0's. In addition, let *use_dim*[ ] be a binary string with exactly k 1's. The image of the embedding contains the mesh nodes that match the pattern $m_1^* \dots m_{n-1}^*$, where:

$$m_i^* = \begin{cases} origin[i], & \text{if } use\_dim[i] = 0 \\ origin[i] \text{ or } origin[i] + 1, & \text{if } use\_dim[i] = 1 \end{cases} \quad (3)$$

This image is available if $k \leq n-1$, since: 1) the width of $M(n-1)$ along any of its dimensions is at least 2, which guarantees that the range selected for the coordinates of the image nodes exists, and 2) at least k different mesh coordinates are needed, which is satisfied when $k \leq n-1$. To show that Alg. 2 embeds $Q(k)$ into $M(n-1)$ with load 1 and dilation 1, it suffices to note that: 1) each node $q \in V(Q(k))$ has a unique image node in $V(M(n-1))$, and 2) if q, q_i are adjacent in $Q(k)$, then their respective image nodes m, m_i are adjacent in $M(n-1)$. □

Algorithm 2 *(Embedding $Q(k)$ onto $M(n-1)$):*

```
cube_to_mesh (int k, n, q[ ], m[ ], origin[ ], use_dim[ ])
{ int i, i_cube = 1;
  for (i = 1; i < n; i = i + 1) {
     m[i] = use_dim[i] × q[i_cube] + origin[i];
     if (use_dim[i] == 1) i_cube = i_cube + 1; } }
```

Alg. 2 uses only a limited range of the coordinates available in $M(n-1)$, producing expansion $n!/2^k$. With multiple calls to Alg. 2 and a proper selection of arguments *origin*[ ] and *use_dim*[ ], one can embed multiple image-disjoint copies of

Appears in *Proceedings of the IEEE 15th Annual International Phoenix Conference on Computers and Communications, Phoenix, AZ, March 27-29, 1996, pp. 115-122.*

$Q(k)$ into $M(n-1)$. This is exactly the basis for the packing techniques we present in this section. Naturally, our ultimate goal is to pack hypercubes into $S(n)$, which can be accomplished with Alg. 1.

4.3 Symmetric fixed-sized packings

In this subsection, we present a symmetric fixed-sized packing P^f which embeds the disjoint union $U^f = \bigcup_{j=0}^{p_k-1} Q_j(k)$ into $S(n)$ with load 1 and base dilation 3, for some $k \in [\lfloor n/2 \rfloor, n-1]$. To make our discussion simpler, we define $t = n-k$. Thus, $t \in [1, \lfloor (n+1)/2 \rfloor]$.

Theorem 1 *For $1 \le t \le \lfloor (n+1)/2 \rfloor$, there is a symmetric fixed-sized packing P^f which embeds the disjoint union $U^f = \bigcup_{j=0}^{p_{n-t}-1} Q_j(n-t)$ into $S(n)$ with load $\lambda(P^f) = 1$, base dilation $d_{base}(P^f) = 3$, and expansion $X(P^f)$, where*

$$p^f_{n-t} = \frac{\lfloor \frac{n}{2} \rfloor ! \cdot \lfloor \frac{n-1}{2} \rfloor ! \cdot t}{2^{t-1}} \cdot \binom{2t-1}{t-1} \quad \text{and} \tag{4}$$

$$X(P^f) = \frac{n}{t \cdot 2^{n-2t+1}} \cdot \binom{n-1}{\lfloor n/2 \rfloor} \cdot \binom{2t-1}{t-1}^{-1} \tag{5}$$

Proof: Noting that the width of $M(n-1)$ along its i^{th} dimension is $s_i = i+1$, we refer to i as an even-sized dimension if s_i is even, and as an odd-sized dimension if s_i is odd. $M(n-1)$ has $\lfloor n/2 \rfloor$ even-sized dimensions and $\lfloor (n-1)/2 \rfloor$ odd-sized dimensions. We partition $M(n-1)$ into: 1) slices of width 1 along the first $t-1$ odd-sized dimensions of $M(n-1)$, and 2) slices of width 2 along all other dimensions of $M(n-1)$.

This partitioning process produces p^f_{n-t} $(n-t)$-dimensional induced submeshes, which we denote by $M_j(n-1)[m_1^* \ldots m_{n-1}^*]$, and index with $0 \le j < p^f_{n-t}$. Each of these submeshes contain all of the nodes of $M(n-1)$ whose labels match the pattern $m_1^* \ldots m_{n-1}^*$, where: 1) for even $i \le 2t-2$, $m_i^* = m_i$, where m_i is an invariant such that $0 \le m_i \le i$, and 2) for odd i or even $i > 2t-2$, either $m_i^* = 2\widehat{m}_i$ or $m_i^* = 2\widehat{m}_i + 1$, where $\widehat{m}_i$ is an invariant such that $0 \le \widehat{m}_i < \lfloor (i-1)/2 \rfloor$.

Due to the partitioning process, these submeshes are disjoint. Each induced submesh has width 2 along its $(n-t)$ dimensions, and can host a copy of $Q(n-t)$ with load 1, dilation 1, which follows from Lemma 1. To embed a copy of $Q(n-t)$ into an induced submesh $M_j(n-1)[m_1^* \ldots m_{n-1}^*]$, one can use Alg. 2 with arguments:

$$use\_dim[i] = \begin{cases} 0, & \text{for even } i \le 2t-2 \\ 1, & \text{otherwise,} \end{cases} \quad \text{and} \tag{6}$$

$$origin[i] = \begin{cases} m_i, & \text{if } use\_dim[i] = 0, \\ 2\widehat{m}_i, & \text{if } use\_dim[i] = 1, \end{cases} \tag{7}$$

for some m_i, $\widehat{m}_i$ such that $0 \le m_i \le i$ and $0 \le \widehat{m}_i < \lfloor (i-1)/2 \rfloor$.

We denote the number of usable slices produced by partitioning $M(n-1)$ along its i^{th} dimension by ν_i, where:

$$\nu_i = \begin{cases} i+1, & \text{for even } i \le 2t-2 \\ \lfloor (i+1)/2 \rfloor, & \text{otherwise} \end{cases} \tag{8}$$

The number of induced submeshes produced by the partitioning process equals the number of packed hypercubes, and is given by:

$$p^f_{n-t} = \prod_{i=1}^{n-1} \nu_i = \frac{\lfloor \frac{n}{2} \rfloor ! \cdot \lfloor \frac{n-1}{2} \rfloor ! \cdot t}{2^{t-1}} \cdot \binom{2t-1}{t-1} \tag{9}$$

Alg. 2 produces the same dimension assignment for each copy of $Q(n-t)$, which characterizes a symmetric packing. If we now embed $M(n-1)$ into $S(n)$ using Alg. 1, a packing with load 1 and base dilation 3 results. To complete the proof, we note that the expansion of the packing can be obtained by direct application of Eq. 1. □

An algorithm that symmetrically packs $Q(n-t)$ into $S(n)$ is given in [9], and is not presented here due to space constraints.

The partitioning technique described in the proof of Theor. 1 discards $1/(i+1)$ of $M(n-1)$ along each odd-sized dimension $i > 2t-2$, which poses a restriction on the expansion ratios that can be achieved. Each unused slice has width 1 along dimension i, and is in fact an $(n-2)$-dimensional submesh of size $2 \times 3 \times \ldots \times i \times (i+2) \times \ldots \times n$. For even $i > 2t-2$, we denote the submesh discarded along dimension i by $M(n-1)[m_i = i]$ (i.e., $M(n-1)[m_i = i]$ is the induced submesh formed by all nodes $m_1 \ldots m_i \ldots m_{n-1} \in V(M(n-1))$, such that $m_i = i$).

Discarded submeshes occur along the last $\lfloor (n+1)/2 \rfloor - t$ odd-sized dimensions i of $M(n-1)$. For $t = \lfloor (n+1)/2 \rfloor$ (or $k = \lfloor n/2 \rfloor$), there are no discarded submeshes. This characterizes a symmetric, fixed-sized packing of $Q(\lfloor n/2 \rfloor)$ into $S(n)$ with load 1, base dilation 3, and expansion 1.

Because discarded submeshes are $(n-2)$-dimensional, they cannot be used to pack additional hypercubes $Q(k)$ with the desired load and base dilation when $k = n-1$, which follows from Lemma 1. However, we can use such submeshes if $\lfloor n/2 \rfloor < k < n-1$, which produces an *asymmetric* fixed-sized packing.

4.4 Asymmetric fixed-sized packings

In this subsection, we present an asymmetric fixed-sized packing P^{f+} which embeds the disjoint union $U^{f+} = \bigcup_{j=0}^{p_k^{f+}-1} Q_j(k)$ into $S(n)$ with load 1 and base dilation 3, for some $k \in [\lfloor n/2 \rfloor + 1, n-2]$. As in the symmetric case, we define $t = n-k$.

Theorem 2 *For $2 \le t \le \lfloor (n-1)/2 \rfloor$, there is an asymmetric fixed-sized packing P^{f+} which embeds the disjoint union $U^{f+} = \bigcup_{j=0}^{p^{f+}_{n-t}-1} Q_j(n-t)$ into $S(n)$ with load $\lambda(P^{f+}) = 1$, base dilation $d_{base}(P^{f+}) = 3$, and expansion $X(P^{f+})$, where*

$$p^{f+}_{n-t} = (1+B) \cdot \frac{\lfloor \frac{n}{2} \rfloor ! \cdot \lfloor \frac{n-1}{2} \rfloor ! \cdot t}{2^{t-1}} \cdot \binom{2t-1}{t-1}, \tag{10}$$

Appears in *Proceedings of the IEEE 15th Annual International Phoenix Conference on Computers and Communications, Phoenix, AZ, March 27-29, 1996, pp. 115-122.*

$$X(P^{f+}) = \frac{n \cdot (1+B)^{-1}}{t \cdot 2^{n-2t+1}} \cdot \binom{n-1}{\lfloor n/2 \rfloor} \cdot \binom{2t-1}{t-1}^{-1}, \quad (11)$$

$$\text{and} \quad B = \frac{t-1}{2t-1} \sum_{h=t}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{h} \quad (12)$$

Proof: We first pack p^f_{n-t} $Q(n-t)$'s into $S(n)$ as described in the proof of Theor. 1, which produces discarded submeshes $M(n-1)[m_i = i]$ for every even $i > 2t-2$. Because the dimensions of interest are odd-sized (i.e., i is even), we define $h = i/2$. For each h such that $t \leq h \leq \lfloor (n-1)/2 \rfloor$, let $p^f_{n-t}\langle 2h \rangle$ denote the number of $Q(n-t)$'s that can be packed into $M(n-1)[m_{2h} = 2h]$. We partition $M(n-1)[m_{2h} = 2h]$ into: 1) slices of width 1 along the first $t-2$ odd-sized dimensions of $M(n-1)[m_{2h} = 2h]$, and 2) slices of width 2 along all other dimensions of $M(n-1)[m_{2h} = 2h]$. This partitioning method produces $p^f_{n-t}\langle 2h \rangle$ $(n-t)$-dimensional submeshes of width at least 2 along any dimension, where:

$$p^f_{n-t}\langle 2h \rangle = \frac{t-1}{h \cdot (2t-1)} \cdot p^f_{n-t} \quad (13)$$

Partitioning $M(n-1)[m_{2h} = 2h]$ as shown above allows us to pack $p^f_{n-t}\langle 2h \rangle$ additional copies of $Q(n-t)$ into $M(n-1)$, with load 1 and dilation 1. Note, however, that the first p^f_{n-t} $Q(n-t)$'s do not have any of their links mapped onto dimension $(2t-2)$ links of $M(n-1)$. The $p^f_{n-t}\langle 2h \rangle$ extra copies, however, use dimension $(2t-2)$ links of $M(n-1)$. Hence, the resulting packing is asymmetric.

We can use the technique just described for all induced submeshes $M(n-1)[m_{2h} = 2h]$, for which $t \leq h \leq \lfloor (n-1)/2 \rfloor$. This produces a total of $p^{f+}_{n-t} = p^f_{n-t} + \sum_{h=t}^{\lfloor (n-1)/2 \rfloor} p^f_{n-t}\langle 2h \rangle$ packed copies of $Q(n-t)$. The theorem follows. □

4.5 Results on fixed-sized packings

Fig. 1 depicts expansion ratios produced by our fixed-sized packing techniques. Note that by reducing the size of the hypercubes being packed, one achieves smaller expansion. For $n = 9$ and $n = 10$, for example, the expansion of our symmetric packings drops from 2.46 to 1.13 as we vary t from 1 to 4. Asymmetry also proves to be an efficient technique to achieve denser packings, resulting in an expansion of at most 1.20 among all asymmetric packings shown in Fig. 1.

4.6 Multiple-sized packings

In this subsection, we present an asymmetric multiple-sized packing P^m which embeds the disjoint union $U^m = \bigcup_{k=\lfloor n/2 \rfloor}^{n-1} \bigcup_{j=0}^{p^m_k - 1} Q_j(k)$ into $S(n)$ with load 1, base dilation 3, and expansion 1. P^m supports hypercube tasks with different node allocation requirements, and guarantees 100% utilization of $S(n)$, $\forall n$.

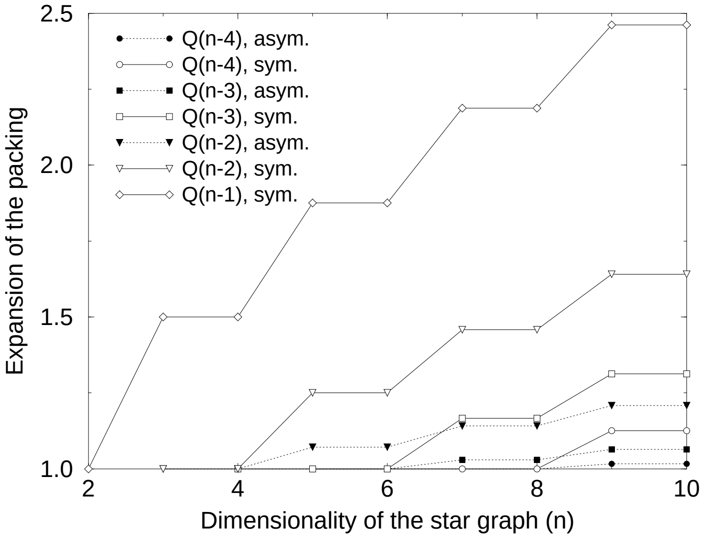

Figure 1: Expansion ratios of packings of $Q(n-t)$ into $S(n)$

The technique used to construct P^m can be summarized as follows. Initially, we pack $Q(n-1)$'s symmetrically into $S(n)$ as described in the proof of Theor. 1. Using the submeshes that are left after this step, we pack $Q(n-2)$'s asymmetrically. This process continues with asymmetric packings of $Q(n-3), Q(n-4), \ldots, Q(\lfloor n/2 \rfloor)$, always using in each step nodes of $M(n-1)$ that were not used in the previous steps. The resulting packing uses all of the nodes in $S(n)$.

Theorem 3 *There is an asymmetric multiple-sized packing* P^m *which embeds the disjoint union* $U^m = \bigcup_{k=\lfloor n/2 \rfloor}^{n-1} \bigcup_{j=0}^{p^m_k - 1} Q_j(k)$ *into* $S(n)$ *with load* $\lambda(P^m) = 1$, *base dilation* $d_{base}(P^m) = 3$, *and expansion* $X(P^m) = 1$, *where*

$$p^m_{n-1} = \left\lfloor \frac{n}{2} \right\rfloor! \cdot \left\lfloor \frac{n-1}{2} \right\rfloor!, \quad p^m_{n-2} = \sum_{h_1=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{p^m_{n-1}}{h_1}, \quad \ldots,$$

$$p^m_{n-t} = \sum_{h_1=1}^{\lfloor \frac{n-2t+3}{2} \rfloor} \sum_{h_2=h_1+1}^{\lfloor \frac{n-2t+5}{2} \rfloor} \cdots \sum_{h_{t-1}=h_{t-2}+1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{p^m_{n-1}}{h_1 \cdot h_2 \cdots h_{t-1}},$$

$$\ldots, \quad p^m_{\lfloor n/2 \rfloor} = p^m_{n-1} \cdot \left(\prod_{h=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{h} \right) \quad (14)$$

Proof: Omitted due to space constraints. The interested reader is referred to [9] for a proof.

5 Variable-Dilation Embeddings

5.1 Basic techniques

In this section, we describe how a $(k+\ell)$-dimensional hypercube can be embedded with variable dilation into $S(n)$,

Appears in *Proceedings of the IEEE 15th Annual International Phoenix Conference on Computers and Communications, Phoenix, AZ, March 27-29, 1996, pp. 115-122.*

$\ell > 0$. As described in Sec. 2, a variable-dilation embedding $W : Q(k+\ell) \mapsto S(n)$ is produced by grouping 2^ℓ copies of $Q(k)$, embedded into $S(n)$ by a packing $P : U \mapsto S(n)$. W transforms P into a packing $P' : U' \mapsto S(n)$, where $U' = U - U_{0,k+\ell} + Q_0(k+\ell)$ and $U_{0,k+\ell} = \bigcup_{j=j_0}^{j_0+2^\ell-1} Q_j(k)$. Because P' embeds $Q(k+\ell)$ into $S(n)$ with dilation $d(P'_{0,k+\ell}) > d_{base}(P)$, packings containing at least one variable-dilation embedding are not template packings. As we shall see, the *average dilation* $d_{aver}(P'_{0,k+\ell})$ of the embedding of $Q_0(k+\ell)$ into $S(n)$ can often be made close to $d_{base}(P)$.

One particularity of our variable-dilation embedding techniques is that the copies of $Q(k)$ needed to compose one $Q(k+\ell)$ (namely, $Q_{j_0}(k)$ through $Q_{j_0+2^\ell-1}(k)$) must have been packed symmetrically by P into $S(n)$. Formally, P should map all dimension a links of $E(U_{0,k+\ell})$ to dimension b links of $E(M(n-1))$, where $1 \le a \le k$ and $1 \le b < n$. This requirement can be met by any template packing P containing a sufficiently large group of symmetrically packed $Q(k)$'s. Hence, even in the case of asymmetric template packings, one will often find one or more usable groups of packed $Q(k)$'s.

To keep our discussion short, we derive the main results of this section for the case $k = n-1$. In Subsec. 5.4, we give examples of variable-dilation embeddings that are formed from $Q(k)$'s for which $k < n-1$.

Our variable-dilation embeddings of $Q(n-1+\ell)$ into $S(n)$ are supported by two template packings that were presented in Sec. 4, namely: 1) the fixed-sized packing of $Q(n-1)$ into $S(n)$, which we denote by P^f, and 2) the multiple-sized packing of $Q(k)$ into $S(n)$ ($\lfloor n/2 \rfloor \le k \le n-1$), which we denote by P^m. Both P^f and P^m contain $\lfloor \frac{n}{2} \rfloor! \cdot \lfloor \frac{n-1}{2} \rfloor!$ symmetrically packed $Q(n-1)$'s, which limits the number of additional hypercube dimensions that can be produced by a variable-dilation embedding of $Q(n-1+\ell)$ into $S(n)$ to:

$$\ell \le \left\lfloor \log_2 \left(\left\lfloor \frac{n}{2} \right\rfloor ! \cdot \left\lfloor \frac{n-1}{2} \right\rfloor ! \right) \right\rfloor \qquad (15)$$

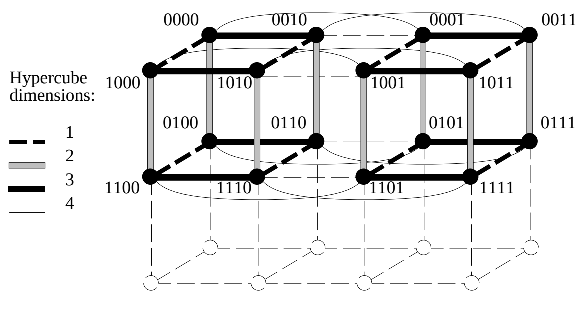

Figure 2: A variable-dilation embedding of $Q(4)$ into $M(3)$

Fig. 2 depicts an example of the technique we will be describing in this section. $Q(4)$ is embedded into $M(3)$ by grouping two packed $Q(3)$'s, which produces dilation 2, average dilation 1.25, and dilation vector $[1,1,1,2]$. If we now embed $M(3)$ into $S(4)$ using Alg. 1, the corresponding embedding of $Q(4)$ into $S(4)$ has dilation 4, average dilation 3.25, and dilation vector $[3,3,3,4]$, which is justified by the following lemma:

Lemma 2 *Let ω, $\omega_{i,\theta} \in V(M(n-1))$ be a pair of nodes separated by θ links along the i^{th} dimension of $M(n-1)$, where $1 \le i < n$ and $1 \le \theta \le i$. Alg. 1 produces an embedding of $M(n-1)$ into $S(n)$, such that the images of ω and $\omega_{i,\theta}$ are connected by a path containing at most $\theta + 2$ links.*

Proof: Omitted due to space constraints. The interested reader is referred to [9] for a proof.

Theorem 4 *Let $f(x,y) = x(y+1) - 2^{x+1} + 2$, and let n, h, and ℓ be integers such that $n \ge 4$, $2 \le h \le \lfloor \log_2 n \rfloor$, and $f(h-1,n) < n-1+\ell \le f(h,n)$. There is a variable-dilation embedding W of $Q(n-1+\ell)$ into $S(n)$, with load $\lambda(W) = 1$, dilation along dimension i of $Q(n-1+\ell)$ $d_i(W)$, average dilation $d_{avr}(W)$, dilation $d(W)$, and expansion $X(W)$, where:*

$$d_i(W) = \begin{cases} 3, & \text{if } 0 < i \le f(1,n) \\ 4, & \text{if } f(1,n) < i \le f(2,n) \\ 6, & \text{if } f(2,n) < i \le f(3,n) \\ \vdots & \vdots \\ \frac{2^e}{2} + 2, & \text{if } f(e-1,n) < i \le f(e,n),\ 1 \le e < h \\ \vdots & \vdots \\ \frac{2^h}{2} + 2, & \text{if } f(h-1,n) < i \le n-1+\ell, \end{cases}$$

$$d_{avr}(W) = \frac{\sum_{i=1}^{n-1+\ell} d_i(W)}{n-1+\ell}, \quad d(W) = \frac{2^h}{2} + 2,$$

$$\text{and } X(W) = \frac{n!}{2^{n-1+\ell}} \qquad (16)$$

Proof: Omitted due to space constraints. The interested reader is referred to [9] for a proof.

We now present an algorithm that produces a variable-dilation embedding of $Q(n-1+\ell)$ into $S(n)$. Such an embedding has the properties specified in Theor. 4.

Algorithm 3 *(Embedding of $Q(n-1+\ell)$ into $S(n)$):*

```
var_embed_cube (int n, ℓ, q[ ], p[ ])
{ int i, m[ ], last;
  for (i = 1; i < n; i = i + 1) m[i] = 0;
  for (e = 1; f(e − 1, n) < (n − 1 + ℓ); e = e + 1)
    last = min(f(e, n), n − 1 + ℓ);
    for (i = f(e − 1, n) + 1; i ≤ last; i = i + 1)
      m[i − f(e − 1, n) + 2^e − 2] + = 2^(e−1) q[i];
  mesh_to_star (n, m[ ], p[ ]); }
int f(int x, y)
{ return(x(y + 1) − 2^(x+1) + 2); }
```

5.2 Advanced techniques

Let $Q(k_a)$ denote the largest hypercube that can be embedded into $S(n)$ via Alg. 3. Due to the restrictions $2 \le h \le \lfloor \log_2 n \rfloor$ and $f(h-1,n) < n-1+\ell \le f(h,n)$ in Theor. 4, we have $k_a = f(\lfloor \log_2 n \rfloor, n) = \lfloor \log_2 n \rfloor (n+1) - 2^{\lfloor \log_2 n \rfloor + 1} + 2$. Accordingly, let $Q(k_b)$ denote the largest hypercube that can

Appears in *Proceedings of the IEEE 15th Annual International Phoenix Conference on Computers and Communications, Phoenix, AZ, March 27-29, 1996, pp. 115-122.*

be embedded with load 1 and variable dilation into $S(n)$, considering only the availability of $Q(n-1)$'s that are produced by either of the packings P^f or P^m. From Eq. 15, we have $k_b = n - 1 + \lfloor \log_2 \left(\lfloor \frac{n}{2} \rfloor ! \cdot \lfloor \frac{n-1}{2} \rfloor ! \right) \rfloor$. Finally, let $Q(k_{max})$ denote the largest hypercube that can be embedded with load 1 into $S(n)$, where $k_{max} = \lfloor \log_2(n!) \rfloor$. We note that an embedding of $Q(k_{max})$ into $S(n)$ can be obtained trivially by a random one-to-one mapping algorithm. Such an approach, however, may result in a dilation of up to $\Phi(S(n))$, where $\Phi(S(n)) = \lfloor 3(n-1)/2 \rfloor$ is the diameter of $S(n)$.

Table 1 lists values of k_a, k_b, and k_{max} for star graphs of practical size. Note that Alg. 3 matches the upper limit k_b, for $4 \le n \le 6$, and produces a maximum dimensionality k_a that is one less than the upper limit k_b, for $7 \le n \le 10$.

$S(n)$	$S(4)$	$S(5)$	$S(6)$	$S(7)$	$S(8)$	$S(9)$	$S(10)$
k_a	4	6	8	10	13	16	19
k_b	4	6	8	11	14	17	20
k_{max}	4	6	9	12	15	18	21

Table 1: Upper limits k_a, k_b and k_{max}

Reference [9] discusses how a load 1, variable-dilation embedding of $Q(k_b)$ into $S(n)$ can be constructed from packed $Q(n-1)$'s, for $7 \le n \le 10$. Dilation vectors for embeddings of $Q(k_b)$ into $S(n)$, for $7 \le n \le 10$, are given in Table 2.

We note that, for $6 \le n \le 10$, the maximum hypercube dimensionality achieved by our variable-dilation embeddings is still one less than k_{max}. To produce an additional hypercube dimension via a variable-dilation embedding, one needs more $Q(n-1)$'s than those produced by packings P^f and P^m. Additional $Q(n-1)$'s can be obtained by grouping smaller hypercubes that are available in P^m, which is discussed in [9]. The details of how these $Q(n-1)$'s can compose $Q(k_{max})$, however, are beyond the scope of this paper.

5.3 Average dilation

Fig. 3 depicts the average dilation d_{avr} produced by the variable-dilation embeddings presented in Subsecs. 5.1 and 5.2. We consider the cases $3 \le k \le 20$ and $4 \le n \le 10$, which correspond respectively to hypercubes of sizes 8 through 1,048,576, and star graphs of sizes 24 through 3,628,800.

As explained in Sec. 2, parallel algorithms that employ a limited amount of hypercube dimensions at any given step may benefit from the smaller average dilation produced by variable-dilation embeddings. Moreover, improved performance can be obtained by reassigning hypercube dimensions prior to the embedding into $S(n)$, which is possible due to the symmetry properties of $Q(k)$. Namely, one can relabel the hypercube nodes, such that in the final embedding into $S(n)$ the most frequently used hypercube dimensions have the smallest dilation.

5.4 Compound var.-dilation embeddings

The variable-dilation embeddings discussed earlier in this section are formed by grouping packed $Q(n-1)$'s. In what follows, we give an example that illustrates how this concept can also be adopted in the case of smaller packed hypercubes. Moreover, we use the example to demonstrate the inherent flexibility that results from combining our multiple-sized packing and variable-dilation embedding techniques.

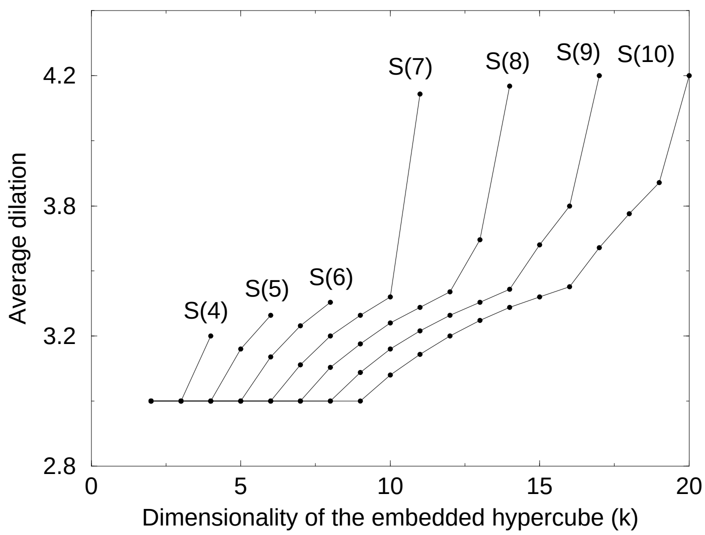

Figure 3: Average dilation of embeddings of $Q(k)$ into $S(n)$

We consider the template multiple-sized packing P^m presented in Subsec. 4.6. Although P^m is an asymmetric packing, it uses a partitioning technique that produces groups of symmetrically packed $Q(k)$'s [9]. These groups of $Q(k)$'s can form variable-dilation embeddings into $S(n)$.

Table 3 lists a few among the many possible transformations that can be produced by applying a series of variable-dilation embeddings to P^m, for the case $n = 8$ (similar tables can be constructed for other values of n). Quantities marked with an * in Table 3 are obtained via variable-dilation embedding techniques. Details of how such quantities are computed are given in [9]. Note that all multiple-sized packings shown in Table 3 have expansion 1.

Packing	P^m	$P^{m,a}$	$P^{m,b}$	$\cdots$	$P^{m,y}$	$P^{m,z}$
# of $Q(4)$*'s*	24	-	-	$\cdots$	-	-
# of $Q(5)$*'s*	144	12*	-	$\cdots$	-	-
# of $Q(6)$*'s*	264	72*	6*	$\cdots$	-	-
# of $Q(7)$*'s*	144	132*	36*	$\cdots$	1*	1*
# of $Q(8)$*'s*	-	72*	66*	$\cdots$	3*	3*
# of $Q(9)$*'s*	-	-	36*	$\cdots$	3*	3*
# of $Q(10)$*'s*	-	-	-	$\cdots$	3*	3*
# of $Q(11)$*'s*	-	-	-	$\cdots$	3*	3*
# of $Q(12)$*'s*	-	-	-	$\cdots$	3*	1*
# of $Q(13)$*'s*	-	-	-	$\cdots$	2*	1*
# of $Q(14)$*'s*	-	-	-	$\cdots$	-	1*

Table 3: Some mult.-sized packings of hypercubes into $S(8)$

6 Comparison with related work

Research on embedding hypercubes into star graphs was pioneered by Nigam, Sahni, and Krishnamurthy, who proposed dilation 2, 3, and 4 embeddings of $Q(k)$ into $S(n)$ [6]. Table 4 lists the largest $Q(k)$'s that can be embedded into $S(n)$

Appears in *Proceedings of the IEEE 15th Annual International Phoenix Conference on Computers and Communications, Phoenix, AZ, March 27-29, 1996, pp. 115-122.*

Embedding (W)	*Dilation vector* ($\overline{d(W)}$)	*Avr. dilation* ($d_{avr}(W)$)	*Dilation* ($d(W)$)
$Q(11) \mapsto S(7)$	$[3,3,3,3,3,3,4,4,4,8,8]$	4.18	8
$Q(14) \mapsto S(8)$	$[3,3,3,3,3,3,3,4,4,4,4,6,8,8]$	4.21	8
$Q(17) \mapsto S(9)$	$[3,3,3,3,3,3,3,3,4,4,4,4,4,6,6,8,8]$	4.25	8
$Q(20) \mapsto S(10)$	$[3,3,3,3,3,3,3,3,3,4,4,4,4,4,4,6,6,6,8,8]$	4.25	8

Table 2: Variable-dilation embeddings of $Q(k_b)$ into $S(n)$, $7 \leq n \leq 10$

with dilation 4, using the techniques of Nigam et al., for $4 \leq n \leq 10$ [6]. Dilation 4 embeddings were chosen because they produce the smallest expansion ratios among the embeddings presented in [6]. Also listed in Table 4 are the corresponding average dilation and expansion produced by our techniques.

Our variable-dilation embedding techniques are an interesting alternative to the dilation 4 embeddings of [6]. For $4 \leq n \leq 10$, we achieve an average dilation ranging from 3.25 to 3.95. Our techniques produce dilation 4, for the cases $4 \leq n \leq 7$, and dilation 6, for the cases $8 \leq n \leq 10$. Note also that, for the cases $8 \leq n \leq 10$, dilation 6 is produced only along: 1) dimension 13 links of $Q(13)$, 2) dimension 15 and 16 links of $Q(16)$, and 3) dimension 17, 18, and 19 links of $Q(19)$.

Embedding	*Dilation (Nigam et al.)*	*Aver. dil. (this paper)*	*Expan. (Nigam et al.)*	*Expan. (this paper)*
$Q(4) \mapsto S(4)$	4.00	3.25	1.50	1.00
$Q(6) \mapsto S(5)$	4.00	3.33	1.88	1.00
$Q(8) \mapsto S(6)$	4.00	3.38	2.81	1.00
$Q(10) \mapsto S(7)$	4.00	3.60	4.92	1.00
$Q(13) \mapsto S(8)$	4.00	3.77	4.92	1.00
$Q(16) \mapsto S(9)$	4.00	3.88	5.54	1.00
$Q(19) \mapsto S(10)$	4.00	3.95	6.92	1.00

Table 4: Comparison with related work

If we consider solely the embedding listed at the left of each row in Table 4, then clearly the expansion ratios resulting from the techniques of [6] and the techniques presented in this paper should be equal. However, as discussed in Secs. 4 and 5, our multiple-sized packings achieve 100% utilization of $S(n)$, meaning that any node of the star graph not used in the embeddings listed in Table 4 can still be used to embed some other hypercube. This certainly allows an efficient use of the star graph, and hence we compute our expansion ratios within the context of a packing (see Eq. 1). From this viewpoint, our optimal expansion (i.e., 1) is always smaller than that achieved in [6].

7 Conclusion

This paper presented novel techniques for packing hypercubes into star graphs, which achieve small expansion and dilation. In particular, the expansion of our multiple-sized packings is optimal. Variable-dilation embeddings resulting from connecting packed $Q(n-1)$'s into $S(n)$ demonstrated the possibility of embedding large hypercubes into the star graph, with corresponding small expansion while maintaining a small dilation on the average. Our techniques can provide the required support for node allocation and task migration strategies in applications where $S(n)$ must handle a workload of parallel algorithms originally devised for the hypercube.

References

[1] S. B. Akers, D. Harel, and B. Krishnamurthy, "The Star Graph: An Attractive Alternative to the n-Cube," *Proc. Int'l Conf. on Parallel Processing*, 1987, pp. 393-400.

[2] J. P. Hayes and T. Mudge, "Hypercube Supercomputers," *Proceedings of the IEEE*, Vol. 77, No. 12, December 1989, pp. 1829-1841.

[3] F. T. Leighton, *Introduction to Parallel Algorithms and Architectures: Arrays · Trees · Hypercubes*, Morgan Kauffmann Publishers, San Mateo, California, 1992.

[4] A. Menn and A. K. Somani, "An Efficient Sorting Algorithm for the Star Graph Interconnection Network," *Proc. Int'l Conf. on Parallel Processing*, 1990, Vol. 3, pp. 1-8.

[5] P. Fragopoulou and S. G. Akl, "A Parallel Algorithm for Computing Fourier Transforms on the Star Graph," *IEEE Transactions on Parallel and Distributed Systems*, Vol. 5, No. 5, May 1994, pp. 525-531.

[6] M. Nigam, S. Sahni, and B. Krishnamurthy, "Embedding Hamiltonians and Hypercubes in Star Interconnection Networks," *Proc. Int'l Conf. Parallel Processing*, 1990, pp. 340-343.

[7] S. Latifi, "Task Allocation in the Star Graph," *IEEE Transactions on Parallel and Distributed Systems*, Vol. 5, No. 11, November 1994, pp. 1220-1224.

[8] S. Ranka, J.-C. Wang, and N. Yeh, "Embedding Meshes on the Star Graph," *Journal of Parallel and Distributed Computing* 19, 1993, pp. 131-135.

[9] M. M. Azevedo, N. Bagherzadeh and S. Latifi, *Space- and Time-Efficient Packings and Embeddings of Hypercubes into Star Graphs*, Department of Electrical and Computer Engineering, University of California, Irvine, Technical Report ECE 94-11-01, November 1994.