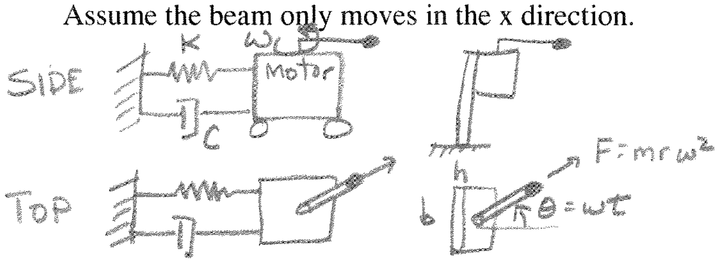
Mechanical Systems Laboratory: Lecture 7
Time and Frequency Response of Second Order Systems

1. A Common Second-Order System: A Mass-Spring-Damper System

In lab next week you will measure how a vibrating beam behaves in the time and frequency domains. The vibrating beam is an example of a system with a mass, some springiness, and some damping. Many physical systems have a mass, some springiness, and some damping, in different proportions. We can describe their behavior with a second order differential equation, and solve the equation to predict responses.

Modeling the Vibrating Beam

Assume the beam only moves in the x direction.



The force caused by the unbalanced load m in the x direction is: $F = mr\omega^2 \sin\theta = mr\omega^2 \sin(\omega t)$

So, we can use the unbalanced load to provide a sinusoidal force input into the beam.

What is K for the beam?

The load-deflection relationship for the beam (from any strength of materials book) is: $x = \dfrac{F\ell^3}{3EI}$

Where:

F = applied load

X = deflections of beam

E = modulus of elasticity

I = area moment of inertia of beam

For a spring: F=Kx, so K = $\dfrac{3EI}{\ell^3}$

So, a simple model is:

$M = M_{motor} + \underbrace{m}_{\text{mass of unbalanced load}} + .236\, M_{beam}$ } because beam has mass; can find using energy methods of vibration analysis

$F = A \sin \omega t \quad A = mr\omega^2$

ω = speed of motor

$C \approx 0$

The differential equation describing this system is:

$$M\ddot{x} = -kx - c\dot{x} + F$$

The transfer function for this system is:

$$(Ms^2 + cs + k)x = F \qquad \frac{x}{F} = H(s) = \frac{1}{Ms^2 + cs + K}$$

Note, for second order systems, we can write the denominator of the transfer function in a general form:

$$s^2 + 2\zeta\omega_n s + \omega_n^2$$

ω_n = "undamped natural frequency"

ζ = damping ratio

For the mass-spring-damper system, find the damping ratio and natural frequency

$$H(s) = \frac{\frac{1}{m}}{s^2 + \frac{c}{m}s + \frac{K}{m}} \Longrightarrow \omega_n = \sqrt{\frac{K}{m}}$$

$$2\zeta\omega_n = \frac{c}{m} \Rightarrow \zeta = \frac{c}{2\sqrt{KM}}$$

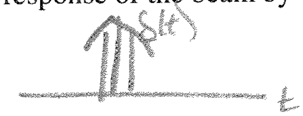
c ← actual damping

$2\sqrt{KM}$ ← "critical" damping

We will see if $\zeta < 1 \Rightarrow$ system is "underdamped" & oscillates

2. How does this system behave in the time domain?

In lab you will measure the transient response of the beam by "twanging" it. How does a system behave when you hit it with an impulse input?



Define $\delta(t) = 0$ for $t \neq 0$

$\int_{-\epsilon}^{\epsilon} \delta(t)\,dt = 1$ for $\epsilon > 0$

$\int_{-\infty}^{\infty} f(t)\delta(t)\,dt = f(0) \Rightarrow$ "sampling property of impulse"

$\delta(t) \longrightarrow \boxed{DEQ} \longrightarrow y(t)$

$\updownarrow \qquad\qquad \updownarrow$

$1 \longrightarrow \boxed{H(s)} \longrightarrow Y(s) = H(s) \times 1 = H(s)$

L.T. of Impulse $\int_0^{\infty} \delta(t) e^{-st}\,dt = e^{-s(0)} = 1$

Thus, the inverse Laplace transform of the transfer function it the impulse response.

What is the impulse response of the vibrating beam?

Use a partial fraction expansion to find the inverse Laplace transform. Basic idea: write the transfer function as the sum of factors that we know how to take the Laplace transform of. Trick to find numerators: multiply by factor, chose s to set factor to zero.

$$H(s) = \frac{\frac{1}{m}}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\frac{1}{m}}{(s-p_1)(s-p_2)} = \frac{A}{s-p_1} + \frac{B}{s-p_2} \Rightarrow h(t) = Ae^{p_1 t} + Be^{p_2 t}$$

The poles of the transfer function are the zeros of the denominator, and they tell us a lot about the way the system behaves, because they became the exponents of exponentials in the time domain.

What are the poles of the vibrating beam?

quadratic eqn $\quad p_i = \dfrac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} \Rightarrow \begin{array}{l} p_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \\ p_2 = \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \end{array}$

$\zeta^2 < 1 \Rightarrow$ poles are imaginary $\Rightarrow$ oscillation

Use partial fraction expansion to find A and B:

$$\left.\frac{\frac{1}{m}(s-p_1)}{(s-p_1)(s-p_2)} = \frac{A(s-p_1)}{(s-p_1)} + \frac{B(s-p_1)}{(s-p_2)}\right|_{s=p_1} \Rightarrow A = \frac{\frac{1}{m}}{p_1 - p_2}$$

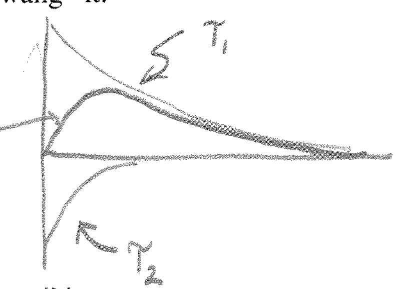
Likewise: $B = \dfrac{\frac{1}{m}}{p_2 - p_1} = -A$

For $\zeta^2 > 1$ the poles are real, and the system does not oscillate when you "twang" it.

$$h(t) = Ae^{p_1 t} + Be^{p_2 t} = A\left(e^{-t/\tau_1} - e^{-t/\tau_2}\right)$$

$\tau_1 = -\frac{1}{p_1} \quad \tau_2 = \frac{-1}{p_2}$

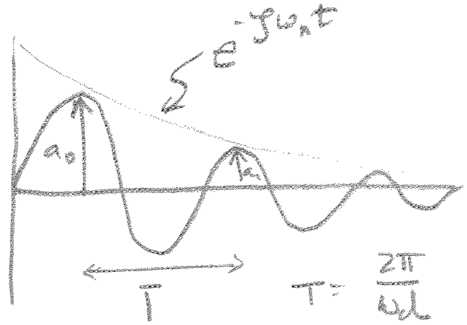
overdamped impulse response



For $\zeta^2 < 1$ the poles are imaginary, and the system oscillates when you "twang" it.

$$p_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} = \zeta\omega_n + \underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d}\,j$$

$$h(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin\omega_d t$$



$T = \dfrac{2\pi}{\omega_d}$

How do you measure damping given the impulse response? One way that you can estimate the damping is by using the "logarithmic damping method"

$$\frac{a_0}{a_n} = e^{-\zeta\omega_n T_n}$$

$$\hat{\delta} \equiv \ln\frac{a_0}{a_n} \Rightarrow \text{solve for } \zeta = \frac{\hat{\delta}}{\sqrt{\hat{\delta}^2 + 4\pi^2 n^2}}$$

calculate $\hat{\delta} = \ln\frac{a_0}{a_n}$ a_0 ← measure, a_n ← measure

n = # of peaks (start counting from zero!)

3. Frequency response of the beam

How does the system behave if you apply a sinusoidal force input to it?

$$\sin\omega t \longrightarrow \boxed{H(s)} \longrightarrow |H(j\omega)|\sin(\omega t + \phi_H(j\omega))$$

Scaling:

Assume $C=0$

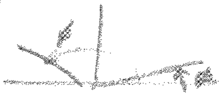
$M\ddot{x} + Kx = F \xrightarrow{\mathcal{L}} (Ms^2+K)X = F$

$$H(s) = \frac{X}{F} = \frac{1}{Ms^2+K} = \frac{1/m}{s^2 + K/m} = \frac{\frac{1}{m}}{s^2+\omega_n^2} \qquad \omega_n = \sqrt{\frac{K}{m}}$$

$$H(j\omega) = \frac{\frac{1}{m}}{(j\omega)^2+\omega_n^2} = \frac{\frac{1}{m}}{\omega_n^2-\omega^2}$$

Phase shift:

$$\phi_H(j\omega) = 0 - \tan^{-1}\frac{0}{\omega_n^2-\omega^2} = \boxed{\begin{cases} 0^\circ & \omega^2<\omega_n^2 \\ 180^\circ & \omega^2>\omega_n^2 \end{cases}}$$



$$\boxed{|H(j\omega)| = \frac{\frac{1}{m}}{|\omega_n^2-\omega^2|}}$$

Plot on a log-log scale (makes curves into lines)

a) Asymptote 1: for $\omega \ll \omega_n$ $|H(j\omega)| \approx \frac{\frac{1}{m}}{\omega_n^2}$ = constant

b) Asymptote 2: for $\omega \gg \omega_n$ $|H(j\omega)| \approx \frac{\frac{1}{m}}{\omega^2} \Rightarrow 20\log|H(j\omega)| = 20\log\frac{1}{m} - 20\log\omega^2$

$$= 20\log\frac{1}{m} - 40\log\omega$$

$$y = b + mx$$

c) $\omega = \omega_n \Rightarrow |H(j\omega)| = \infty$

Resonance

What are some uses of resonance?

the military (star wars)

communication (detecting 1 frequency)

entertainment (musical instruments)

medicine (shattering a kidney stone w/ ultrasound)

the playground (swing set)