1 Preliminaries and Background Material

1.1 Complex Numbers, Polar coordinates

Recall that we define

\[ i = \sqrt{-1} \quad (\text{sometimes we use } 'j' !) \quad (1.1) \]

Now in the diagram above, consider point ‘P’ in the complex plane. We denote the length \( OP = A \), where clearly,

\[ a = A \cos \theta, \quad b = A \sin \theta \quad (1.2) \]

We can use one of the following coordinates:

**Cartesian Coordinates** : \( P = a + ib \) \quad (1.3)

**Polar Coordinate** : \( P = Ae^{i\theta} \) \quad (1.4)
Equations (1.2)-(1.4) imply $e^{i\theta} = \cos\theta + j\sin\theta$, which is the same as the Euler’s’ identity further below. We can use these equations to do basic operations on complex numbers.

**Addition:** let

$$X_1 = A_1 e^{i\omega t}, \quad X_2 = A_2 e^{i(\omega t + \phi)}$$

where $\phi$ is the phase angle between the $X_1$ and $X_2$. Then

$$X_1 + X_2 = (A_1 + A_2 e^{i\phi}) e^{i\omega t} = [(A_1 + A_2 \cos \phi) + iA_2 \sin \phi] e^{i\omega t} = A_3 e^{i(\omega t + \alpha)}$$

where

$$A_3^2 = (A_1 + A_2 \cos \phi)^2 + (A_2 \sin \phi)^2$$

and

$$\tan \alpha = \frac{A_2 \sin \phi}{A_1 + A_2 \cos \phi}$$

**Multiplication:** let

$$X_1 = A_1 e^{i\theta_1}, \quad X_2 = A_2 e^{i\theta_2}$$

then

$$X_1 X_2 = A_1 A_2 e^{i(\theta_1 + \theta_2)}$$

Why? Can you show it?

**Square root, etc.:** From multiplication, we know that

$$(A e^{i\theta})(A e^{i\theta}) = A^2 e^{i2\theta}$$

therefore, we can write

$$(A e^{i\theta})^n = A^n e^{in\theta}$$

where $n$ can be any real number (and not necessarily an integer). For example,

$$(A e^{i\theta})^{\frac{1}{2}} = A^{\frac{1}{2}} e^{i\frac{\theta}{2}}$$

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or more generally

$$(Ae^{i\theta})^{\frac{1}{2}} = A^{\frac{1}{2}} e^{i\frac{\theta}{2}}$$

**Derivatives:** Now suppose that $\theta$ was changing with a constant rate of change, i.e., $\theta = wt$ for some $w$, and point $P$, as a result, is rotating counter clockwise. Then vector $\mathbf{P}$ is changing with time, therefore, it has time derivative (or velocity!), i.e.,

$$\mathbf{P} = Ae^{iwt} \tag{1.5}$$

$$\dot{\mathbf{P}} = iwAe^{iwt} = iw\mathbf{P} \tag{1.6}$$

$$\ddot{\mathbf{P}} = -w^2Ae^{iwt} = -w^2\mathbf{P} \tag{1.7}$$

Finally, think about taking derivative of '$a$' in (1.2). We can write (keep your eyes on the subtle use of chain rule in taking derivatives)

$$\dot{a} = -wAsin\theta, \quad \ddot{a} = -w^2cos\theta \tag{1.8}$$

or we can use the following

$$a = \text{Real}(\mathbf{P}) = \text{Real}(Ae^{iwt})$$

and therefore,

$$\dot{a} = \text{Real}(\dot{\mathbf{P}}) = \text{Real}(iw\mathbf{P}) = -wAsin\theta \tag{1.9}$$

and similarly

$$\ddot{a} = \text{Real}(\ddot{\mathbf{P}}) = \text{Real}(-w^2\mathbf{P}) = -w^2Acos\theta \tag{1.10}$$

We will see in future chapters that this trick can save a great deal of time.
2 Power Series, etc

\[ e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad (2.11) \]

\[ \cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \quad \text{for } \theta \text{ real } \quad (2.12) \]

\[ \sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \quad \text{for } \theta \text{ real } \quad (2.13) \]

2.1 Euler’s Identity, . . .

\[ e^{i\theta} = \cos \theta + i \sin \theta \quad (2.14) \]

\[ e^{-i\theta} = \cos \theta - i \sin \theta \quad (2.15) \]

\[ \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad (2.16) \]

\[ \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = -\frac{i(e^{i\theta} - e^{-i\theta})}{2} \quad (2.17) \]

2.2 Trigonometric Identities

\[ \sin A + \sin B = 2 \sin \left( \frac{A + B}{2} \right) \cos \left( \frac{A - B}{2} \right) \quad (2.18) \]

\[ \cos A + \cos B = 2 \cos \left( \frac{A + B}{2} \right) \cos \left( \frac{A - B}{2} \right) \quad (2.19) \]

\[ \sin(A + B) = \sin A \cos B + \sin B \cos A \quad (2.20) \]

\[ \cos(A + B) = \cos A \cos B - \sin A \sin B \quad (2.21) \]

\[ \sin A \sin B = \frac{1}{2} [-\cos(A + B) + \cos(A - B)] \quad (2.22) \]

\[ \cos A \cos B = \frac{1}{2} [\cos(A + B) + \cos(A - B)] \quad (2.23) \]

\[ \sin A \cos B = \frac{1}{2} [\sin(A + B) + \sin(A - B)] \quad (2.24) \]
2.3 Orthogonality Conditions for Fourier Series

Let \( w = \frac{2\pi}{\tau} \), where \( \tau \) is in seconds (i.e., period of motion), then for all positive integers \( n \) and \( m \) following identities hold:

\[
\int_{-\tau/2}^{\tau/2} \cos nwt \cos mwt \, dt = \begin{cases} 
0 & \text{if } m \neq n \\
\frac{\tau}{2} & \text{if } m = n 
\end{cases} \quad (2.25)
\]

\[
\int_{-\tau/2}^{\tau/2} \sin nwt \sin mwt \, dt = \begin{cases} 
0 & \text{if } m \neq n \\
\frac{\tau}{2} & \text{if } m = n 
\end{cases} \quad (2.26)
\]

\[
\int_{-\tau/2}^{\tau/2} \cos nwt \sin mwt \, dt = \begin{cases} 
0 & \text{if } m \neq n \\
0 & \text{if } m = n 
\end{cases} \quad (2.27)
\]

\[
\int_{-\pi}^{\pi} e^{inx} e^{-inx} \, dx = \begin{cases} 
0 & \text{if } m \neq n \\
2\pi & \text{if } m = n 
\end{cases} \quad (2.28)
\]
3 Fourier Series

Consider a periodic function $x(t)$ with a period of $\tau$; i.e.,

$$x(t) = x(t + \tau) = x(t + 2\tau) = \ldots \text{ for all } t.$$

We define the following:

$$w_1 = \frac{2\pi}{\tau} \text{ fundamental frequency or 'harmonic'}$$

$$w_n = nw_1, \quad n = 2, 3, 4, \ldots \text{ higher harmonics}$$

The basic idea is that the periodic function, $x(t)$, can be represented by a, possibly infinite, sum of its harmonics (i.e., its ‘Fourier Series Representation’):

$$x(t) = \frac{a_0}{2} + a_1 \cos w_1 t + a_2 \cos w_2 t + a_3 \cos w_3 t + \cdots \quad (3.1)$$

$$+ b_1 \sin w_1 t + b_2 \sin w_2 t + b_3 \sin w_3 t + \cdots \quad (3.2)$$

Finding $a_i$’s and $b_i$’s (which are called the Fourier coefficients) is relatively simple. For example, multiply (3.1) by $\cos w_n t$, integrate both sides and use the orthogonality conditions (from the previous section) to get the following

$$a_n = \frac{2}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} x(t) \cos w_n t \, dt, \quad w_n = nw_1, \quad n = 1, 2, 3, \ldots \quad (3.3)$$

$$b_n = \frac{2}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} x(t) \sin w_n t \, dt, \quad w_n = nw_1, \quad n = 1, 2, 3, \ldots \quad (3.4)$$

$$a_0 = \frac{2}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} x(t) \, dt \quad \text{i.e., the average}$$
Next, recall that

\[ A \cos X + B \sin X = \sqrt{A^2 + B^2} \cos (X - \phi) \]  \hspace{1cm} (3.5)

\[ = \sqrt{A^2 + B^2} \sin (X - \phi) \]

where \( \tan \phi = \frac{B}{A} \)

Therefore, we can rewrite the Fourier series representation as

\[ x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n \cos(w_n t - \phi_n), \]  \hspace{1cm} (3.6)

\[ c_n = \sqrt{A^2 + B^2}, \tan \phi = \frac{b_n}{a_n} \]

### 3.1 Complex Representation

We can also use Euler’s identity to provide another form of the Fourier series. Using the (complex) exponential form of cos and sin (e.g., \( \cos x = \frac{e^{ix} + e^{-ix}}{2} \)), we can rewrite (3.1) as

\[
x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ \frac{a_n}{2} e^{iw_n t} + \frac{b_n}{2i} e^{-iw_n t} \right]
\]

\[
= \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{iw_n t} \left( \frac{a_n}{2} + \frac{b_n}{2i} \right) + \sum_{n=1}^{\infty} e^{-iw_n t} \left( \frac{a_n}{2} - \frac{b_n}{2i} \right)
\]

recalling \( w_n = nw_1 \), we can write

\[
x(t) = \sum_{n=-\infty}^{\infty} c_n e^{inw_1 t}, \text{ where } \begin{cases} 
   c_0 = \frac{a_0}{2} \\
   c_n = \frac{a_n}{2} - \frac{ib_n}{2i}, & n > 0 \\
   c_n = \frac{a_n}{2} + \frac{ib_n}{2i}, & n < 0 
\end{cases}
\]

We can simplify a bit more. For example, consider the expression for \( c_n \) for \( n > 0 \): using the definitions of \( a_n \) and \( b_n \) from (3.3) and (3.4), respectively, we get

\[
c_n = \frac{1}{2} (a_n - ib_n) = \frac{1}{2} \left\{ \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} (x(t) \cos nw_1 n - ix(t) \sin nw_1 t) \, dt \right\}
\]

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or
\[ c_n = \frac{1}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} x(t)(\cos nw_1 n - i \sin nw_1 t) \, dt = \frac{1}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} x(t)e^{-iwn t} \, dt \]

Similarly, for \( n < 0 \) we get \( c_n = \frac{1}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} x(t)e^{iwn t} \, dt \). Combining the two expressions, we have the following

\[ x(t) = \sum_{n=-\infty}^{\infty} c_n e^{iwn t}, \quad \text{where} \quad c_n = \frac{1}{\tau} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} x(t)e^{-i|n|wn t} \, dt \quad (3.7) \]

3.2 Odds and Ends

- In general, calculating \( a_n 's \) and \( b_n 's \) is a pain, but certain simplifications can be made. For example, if \( x(t) \) is either ‘odd’ or ‘even’, half of the coefficients are automatically zero (see the textbook, Chapter 5).

- Fourier series has all sorts of applications. The main interest we have in this class, is when the forcing function (which caused the vibration) is periodic. In Chapter 5, we will use Fourier series to breakdown a complicated periodic function into a bunch of nice and simple sines and cosines (Which are handled by the approach we will see in Chapter 4!)

- If you think about it, you will see that there is no need spend your life calculating Fourier series by hand. Particularly when the function \( x(t) \) is a complicated mess, one would calculate the series using a computer.
4 Notes for Chapter 4

4.1 Frequency Response

Consider the differential equation,

\[ m\ddot{x} + c\dot{x} + kx = F(t) = F_o \cos wt. \]  \hfill (4.1)

After taking the Laplace transform, we get

\[ x(s) = \frac{msx(0) + m\dot{x}(0) + cx(0)}{ms^2 + cs + k} + \frac{F(s)}{ms^2 + cs + k} = x_1(s) + x_2(s) \]  \hfill (4.2)

where \( F(s) \) is the Laplace transform of \( F(t) \), i.e.,

\[ F(s) = \frac{F_0s}{s^2 + w^2} \]

With these, we make the following definition

**Definition:** Set \( x(0) = \dot{x}(0) = 0 \).

\[ H(s) = \frac{x(s)}{F(s)} = \frac{1}{ms^2 + cs + k} \]  \hfill (4.3)

i.e., the transfer function is the ratio of the output (i.e., \( x \)) to the input (i.e., \( F \)) in the s-domain, setting all initial condition to zero. Now, going back to (4.2), and taking inverse Laplace, we have

\[ x(t) = x_1(t) + x_2(t). \]

Consider the first term. From the results of chapter 2, we know that

\[ x_1(t) = e^{-\zeta w dt}(X_1 \cos w dt + X_2 \sin w dt) \]

where \( X_1 \) and \( X_2 \) depend on the initial conditions and \( w_d = w_n \sqrt{1 - \zeta^2} \). As long as we have some damping, the exponential terms decays to zero; i.e., \( x_1(t) \to 0 \) as \( t \to \infty \). As a result,

\[ x_{\text{steady state}}(t) = x_{ss}(t) = x_2(t). \]
From now on, let us focus on this steady state solutions (i.e., we either have no initial conditions or we assume that the transients die out fast and we need to study long term behavior). We will try the odd ‘partial fraction expansion’ method

\[ x_2(s) = H(s) F(s) = \frac{1}{ms^2 + cs + k} F_0 s = \frac{As + B}{s^2 + w^2} + \frac{Ds + C}{ms^2 + cs + k} \]  

(4.4)

where \( A, B, C \) and \( D \) are to be found. Now the last term (i.e., \( \frac{Ds + C}{ms^2 + cs + k} \)) results (after inverse Laplace transform) in terms like \( e^{-\zeta w t} (\ldots) \), just as in \( x_1(t) \), which goes to zero as time goes to infinity. Therefore, for the steady state response, we can concentrate on the term \( \frac{As + B}{s^2 + w^2} \). For this, multiply (4.4) by \( s^2 + w^2 \) to get

\[ H(s) F_0 s = (As + B) + H(s)(Ds + c)(s^2 + w^2). \]  

(4.5)

This equation is true for all values of \( s \). In particular, we can use two specific values that let the last term disappear; i.e, \( s = \pm iw \) to get

\[ H(iw) F_0 iw = Awi + B \]  

(4.6)

\[ -H(-iw) F_0 iw = -Awi + B \]  

(4.7)

Solving these two equations (by adding and subtracting them from each other) yields

\[ \begin{cases} B = \frac{1}{2} F_0 wi [H(iw) - H(-iw)] \\ A = \frac{1}{2} F_0 [H(iw) + H(-iw)] \end{cases} \]  

(4.8)

Since the rest of the coefficients of \( H(s) \) are real, we have \( H(-iw) = \overline{H}(iw) \) which results in

\[ \begin{cases} A = F_0 \text{Real}[H(iw)] \\ B = -F_0 w \text{Imag}[H(iw)] \end{cases} \]  

(4.9)

As a result, using (4.8) in (4.4), after a little bit of work, we have

\[ x_{ss}(s) = F_0 \text{Real}[H(iw)] \frac{s}{s^2 + w^2} - F_0 \text{Imag}[H(iw)] \frac{w}{s^2 + w^2} \]  

(4.10)
or, after taking the inverse Laplace

\[ x_{ss}(t) = F_o \text{Real}[H(iw)] \cos wt - F_o \text{Imag}[H(iw)] \sin wt \]  

(4.11)

using the trig. identity

\[ A \cos X - B \sin X = \sqrt{A^2 + B^2} \cos(X + \phi), \quad \tan \phi = \frac{B}{A} \]

we get

\[
\begin{align*}
    x_{ss}(t) &= |H(iw)| F_o \cos(wt + \phi) \\
    \tan \phi &= \frac{\text{Imag}[H(iw)]}{\text{Real}[H(iw)]} \\
\end{align*}
\]

(4.12)

where \(|H(iw)|\) is called the ‘gain’ and \(\phi\) is the phase of the transfer function \(H(s)\).

**NOTE:** \(H(s)\) can be more general, i.e., larger than second order. The algebra would be somewhat more complicated, but the same result, i.e., (4.12).

**SUMMARY:** When the input (i.e., the forcing function) for the (4.1) is a simple sine (or cosine), the steady state response is a sine (or cosine), at the same frequency. The magnitude of the response is the magnitude of the forcing function multiplies by the magnitude of the transfer function of the system at the frequency of the forcing function (i.e., \(|H(iw)|\)). The response will also have a phase difference with the input (which is equal of the angle of \(H(iw)\)). The process can be seen below.

**Step 1:** Write the differential equation of motion

**Step 2:** Take Laplace Transform

**Step 3:** Set initial conditions to zero and find \(H(s)\) in \(x(s) = H(s)F(s)\), where \(F(s)\) is the Laplace transform of the forcing function.

**Step 4:** The steady state gain - magnification - from input to output is \(|H(iw)|\) and the phase is \(\phi\) as define above. (\(w\) is the frequency of the forcing function)
5 Some note on Chapter 5

We start with

\[ m\ddot{x} + c\dot{x} + kx = F(t) \]  \hspace{1cm} (5.1)

In chapter 2, we set the forcing function to be zero, while in chapter 3 we solved the problem when the forcing function was a simple sine or cosine. Here, we study the case where \( F(t) \) has more complicated forms.

5.1 A: \( F(t) \) is periodic

When the forcing function is periodic; i.e., \( F(t) = F(t + \tau) = F(t + 2\tau) \ldots \) for some fixed period \( \tau \), we can use the Fourier series results we studied in Chapter 1. Recall that we can ‘break’ down a nasty looking periodic function as a sum of a sines and cosines; i.e.,

\[ F(t) = \frac{a_0}{2} + a_1 \cos wt + b_1 \sin wt + a_2 \cos 2wt + b_2 \sin 2wt + \cdots \]
\[ = f_1(t) + f_2(t) + f_3(t) + f_4(t) + f_5(t) + \cdots \]  \hspace{1cm} (5.2)

Equation (5.1) thus becomes

\[ m\ddot{x} + c\dot{x} + kx = f_1(t) + f_2(t) + f_3(t) + f_4(t) + f_5(t) + \cdots \]  \hspace{1cm} (5.3)

Next, recall from your differential equation course (M3D, for example) that we can use the ‘superposition’ technique for a linear differential equation. That is, we solve (5.1) as if \( f_i(t) \) is the only forcing function and call the solution to it \( x_i(t) \). Repeating this for \( i = 1, 2, \ldots \) the solution to the actual problem is

\[ x(t) = x_1(t) + x_2(t) + x_3(t) + x_4(t) + x_5(t) \]  \hspace{1cm} (5.4)

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where the \( x_i(t) \)'s satisfy

\[
\begin{align*}
    m \ddot{x}_1 + c \dot{x}_1 + k x_1 &= f_1(t) = \frac{\alpha_o}{2} \\
    m \ddot{x}_2 + c \dot{x}_2 + k x_2 &= f_2(t) = a_1 \cos wt \\
    m \ddot{x}_3 + c \dot{x}_3 + k x_3 &= f_3(t) = b_1 \sin wt \\
    \vdots & \quad \vdots & \quad \vdots & \quad \vdots \quad \vdots
\end{align*}
\]

(5.5)

Each of the equations in (5.5) was solved in chapter 3. See the text book (page 265, equation 4.13) for the complete solution (with all of the bells and whistles). We can summarize as

**Summary:**

- Break down the periodic forcing function, \( F(t) \), into its harmonic components via the Fourier series.

- Solve the simple equations in (5.5) (each with a single harmonic forcing function) and solve for \( x_i(t) \).

- Add all of these \( x_i(t) \) together to get the whole solution.

5.2  \( F(t) \) is not periodic, but \( F(s) \) is obtainable

Next, we consider the forcing functions that are not necessarily periodic (they could be!) and we assume that we can find the Laplace transform of the forcing function; i.e, \( F(s) \) is known. In this case, we simply take the Laplace transform of the equation, find the expression for \( x(s) \) and then take the inverse Laplace transform to get \( x(t) \). That is, first we take the transform of (5.1) to get

\[
(ms^2 + cs + k)x(s) = F(s) + m sx(0) + m \dot{x}(0) + cx(0)
\]
where $F(s)$ is the Laplace transform of $F(t)$. Now we have

\[
x(s) = \frac{1}{ms^2 + cs + k} F(s) + \frac{ms \dot{x}(0) + m \ddot{x}(0) + cr(0)}{ms^2 + cs + k} = H(s) F(s) + H(s) [ms \dot{x}(0) + m \ddot{x}(0) + cr(0)]
\]

where we have used $H(s)$ (i.e., the transfer function) as defined in the previous chapter.

Next, we just go ahead and take the inverse Laplace of $x(s)$; i.e.,

\[
x(t) = \mathcal{L}^{-1}[x(s)]
\]  

If you were interested in only the steady state solution, you would ignore the second term (or set the initial conditions to zero). The only problem is that this can end up being very messy and tedious and is probably useful for a ‘simple’ forcing functions.

### 5.3 General Forcing Function

Now let us consider the most general case: While $F(s)$ may exist, either we can not find or calculate $F(s)$ or it is very tedious and hard to do so. We go back to the previous subsection. For a moment assume that we had $F(s)$, recall from (5.6) that

\[
x(s) = \frac{1}{ms^2 + cs + k} F(s) + = H(s) F(s) + H(s) [ms \dot{x}(0) + m \ddot{x}(0) + cr(0)] = H(s) F(s) + x_T(s)
\]

where we use $x_T$ to denote the component of $x$ that is ‘transient’ and does not play a role in steady state (and is the exponentially decaying functions we saw in Chapter 2). Now, after taking the inverse Laplace transform of this equation, we have

\[
x(t) = x_T(t) + \mathcal{L}^{-1}[H(s) F(s)]
\]  

where $x_T(t)$ is exponentially decaying part. So, we focus on the second part of $x(t)$. Now recall the ‘convolution’ property of the Laplace transform (see Section 4.7 of the test, for example) which says

\[
\mathcal{L} \left[ \int_0^t f_1(t - \tau) f_2(\tau) d\tau \right] = F_1(s) F_2(s)
\]  

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or equivalently
\[
\mathcal{L}^{-1} \left[ F_1(s) F_2(s) \right] \left[ \int_0^t f_1(t - \tau) f_2(\tau) \, d\tau \right]
\]

where \( F_1(s) \) and \( F_2(s) \) are the Laplace transforms of \( f_1(t) \) and \( f_2(t) \), respectively. Now comparing (5.8) and (5.9), we can write
\[
x(t) = x_T(t) + \int_0^t h(t - \tau) F(\tau) d\tau = x_T(t) + \int_0^t F(t - \tau) h(\tau) d\tau \quad (5.10)
\]

(make sure you can show the two integrals in the last equation are equal to one another).

Here we have used the following definitions
\[
F(t) = \mathcal{L}^{-1}[F(s)], \quad \text{the forcing function}
\]

\[
h(t) = \mathcal{L}^{-1}[H(s)], \quad \text{the Impulse Response} \quad (5.11)
\]

Now, equation (5.10) can be computed numerically if one has data for \( F(t) \) (for example, records of wind gusts or ground motion during an earthquake) and does not have to be done analytically.

The last remaining question is regarding the \( h(t) \) - which was defined in (5.11) as the inverse Laplace of the Transfer Function \( H(s) \). In our problem, we have
\[
H(s) = \frac{1}{ms^2 + cs + k} = \frac{1}{m} \frac{s^2 + \frac{c}{m} s + \frac{k}{m}}{s^2 + 2\zeta w_n s + w_n^2}
\]

using the definitions we used for \( \zeta \) and \( w_n \) in chapters 2 and 3 (i.e., \( w_n^2 = \frac{k}{m} \) and \( \frac{c}{m} = 2\zeta w_n \)).

Next, using the Laplace transforms table (e.g., the one in the back of the text), we have
\[
h(t) = \frac{1}{mw_d} e^{-\zeta w_n t} \sin w_d t \quad w_d = w_n \sqrt{1 - \zeta^2} \quad (5.12)
\]
5.4 Closing Remarks

1. $h(t)$ is called the ‘impulse response’ of the system because it is the response - or reaction - of the system (i.e., the mass spring dashpot system) to an ‘impulsive’ force of unit intensity. A unit impulse force is a large sudden force that is applied for a very short duration (such that the integral $\int F \, dt=1$).

2. You can also get $h(t)$ if you release the system with $x(0) = 0$ but $\dot{x}(0) = 1/m$. The resulting $x(t)$ would be exactly the same as $h(t)$ in (5.1) - see text for details.

3. Algorithm: The approach in this subsection can be considered as following this algorithm:

   • get $F(t)$, either in closed form or by data
   • find $h(t)$, basically from (5.11) or (5.12)
   • Calculate the integral, either analytically (by hand or Mathematica, etc) or numerically.

4. The approach here applies to more general systems. Systems with more degrees of freedom (e.g, more than one mass), systems with more forcing functions, etc. All that is required is $F(t)$ and the $h(t)$. 

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6 Notes for Chapter 6

6.1 A Brief Review of Linear Algebra

Consider the systems of equations

\[
\begin{align*}
  a_{11} x_1 + a_{12} x_2 &= 0 \\
  a_{12} x_1 + a_{22} x_2 &= 0
\end{align*}
\]  \hspace{1cm} (6.1)

where \(a_{11}, a_{12}, a_{21}\) and \(a_{22}\) are known constants. We can write this set of equations in the matrix form

\[
Ax = 0
\]

where

\[
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

Now, we recall the following from linear algebra

- If \(\det[A] \neq 0\), then \(A^{-1}\) exists, and therefore,

\[
Ax = 0 \quad \Rightarrow \quad A^{-1}Ax = 0 \quad \Rightarrow \quad \vec{x} = 0
\]

Now, if we hope to find nontrivial (i.e., non zero) solutions to this equation, we need to avoid what just happened, i.e., we need to avoid \(\det[A] \neq 0\)!

- \(\det[A] = 0\), then there exits (many) solutions to the equation. For example, if \(\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\) is a solution, so are \(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\), \(\begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}\), and \(\alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix}\) for any \(\alpha\) (why??)

Now, recall that if \(\det[A] = 0\) then the two equations in (6.1) are not independent. In short, there is really one equation and two unknowns. Therefore, you cannot solve for both...
of the unknowns. The best you can do is to express one of the unknowns in terms of the other one. For example,

\[ x_1 = -\frac{a_{12}}{a_{11}} x_2 \quad \text{or} \quad x_2 = -\frac{a_{11}}{a_{12}} x_1 \]

so if you knew either \( x_1 \) or \( x_2 \), you could find the other one. As a result, the solutions are of the form

\[
\begin{pmatrix}
  x_1 \\
  -\frac{a_{11}}{a_{12}} x_1
\end{pmatrix} = \begin{pmatrix}
  1 \\
  -\frac{a_{11}}{a_{12}}
\end{pmatrix} x_1
\]

or equivalently,

\[
\begin{pmatrix}
  -\frac{a_{12}}{a_{11}} x_2 \\
  x_2
\end{pmatrix} = \begin{pmatrix}
  -\frac{a_{12}}{a_{11}} \\
  1
\end{pmatrix} x_2.
\]

### 6.2 Back to Chapter 5

Suppose that free vibration of a system is modeled by the following differential equations

\[
M\ddot{x}(t) + Kx(t) = 0 \tag{6.2}
\]

where \( M \) and \( K \) are the mass and stiffness matrices \( 2 \times 2 \), respectively, and \( x(t) \) is a \( 2 \times 1 \) vector.

Parallel to the single degree of freedom, we make the following assumption regarding the form of the solution for (6.2) (an assumption that is verified and motivated by Laplace Transform - believe it or not)

\[
x(t) = X \cos(\omega t + \phi) \quad \text{or} \quad X \sin(\omega t + \phi) \quad \tag{6.3}
\]

We need to find \( \omega \) and \( \phi \). Use (6.3) and plug in (6.2) to get

\[
-w^2 M X \cos(\omega t + \phi) + K X \cos(\omega t + \phi) = 0 \quad \Rightarrow \quad (K - w^2 M) X \cos(\omega t + \phi) = 0, \quad \text{for all } t.
\]

Since \( \cos(\omega t + \phi) \) is not zero all the time, (and \( K - w^2 M \) is constant) we must have

\[
(K - w^2 M) X = 0 \tag{6.4}
\]

6–2
Now, $X$ is the magnitude of the motion (recall (6.3)), so we are only interested in nonzero $X$. Remembering the results from the previous section, the problem has non-zero solution if

$$\det[K - w^2 M] = 0$$

which is a determinant of a $2 \times 2$ matrix. Now if you remember how to take determinants (BIG IF!), you would notice that the determinant will have constants, powers of $w^2$ and $w^4$ only (think about this a bit). So it is a second order equation in terms of $w^2$ and we can solve it rather easily; i.e.,

$$\det[K - w^2 M] = 0 \Rightarrow 2 \text{ values for } w^2 : w_1^2, w_2^2$$

We will see later (in Chapter 8) that the values one gets for $w_1^2$ and $w_2^2$ will always be either positive or zero. Now each $w_i$ corresponds to a value of $w$ for which $\det(K - w^2 M)$ becomes zero. Therefore, for these values of $w$ only, we can find nonzero vectors $X$. Since we have two potential values for $w$, we write the solution as

$$\underline{x}(t) = X^{(1)} \cos(w_1 t + \phi_1) + X^{(2)} \cos(w_2 t + \phi_2) \quad (6.5)$$

Note the similarity between the general form of the solution for the 2-degree-of-freedom system and that of the 1-degree of freedom system we saw earlier.

Now let us go back to (6.4). We have two values for $w$ that make the determinant zero (and for other values of $w$ this determinant is not zero). For example, for $w_1$, we have the determinant zero and thus we can have a nontrivial (i.e., nonzero) solution to (6.4), i.e., we can hope to solve for

$$(K - Mw_1^2)X^{(1)} = 0$$

where $X^{(1)}$ is the $X$ solution in (6.4) that corresponds to $w_1$. From the review material, we
know that we cannot solve for $X$ completely, but we can get an answer of the form

$$X^{(1)} = \begin{bmatrix} X_1^{(1)} \\ r^{(1)} X_1^{(1)} \end{bmatrix} = X_1^{(1)} \begin{bmatrix} 1 \\ r^{(1)} \end{bmatrix}$$ (6.6)

Where $X_1^{(1)}$ is unknown, but $r^{(1)}$ depends on the entries of $K$ and $M$ and can be calculated easily. Similarly, we can start with $w_2$ and solve the corresponding $X$ in the form

$$X^{(2)} = \begin{bmatrix} X_1^{(2)} \\ r^{(2)} X_1^{(2)} \end{bmatrix} = X_1^{(2)} \begin{bmatrix} 1 \\ r^{(2)} \end{bmatrix}$$ (6.7)

Using these last two expressions, we get the following form for the general solution

$$\hat{z}(t) = X_1^{(1)} \begin{bmatrix} 1 \\ r^{(1)} \end{bmatrix} \cos (w_1 t + \phi_1) + X_1^{(2)} \begin{bmatrix} 1 \\ r^{(2)} \end{bmatrix} \cos (w_2 t + \phi_2)$$ (6.8)

NOTES:

- Note that there are four unknowns in the general solution (the two $\phi_i$ and the two $X^{(i)}$). See equations in the text for explicit formulas that can be used to obtain these unknowns, given the initial positions and velocities.

- $w_1$ and $w_2$ are called the natural frequencies of the system. They depend on $K$ and $M$ only and not the initial conditions. Also, judging from the general solution, once the system is released from a set of initial conditions, the motion ensuing motion is either a periodic motion with $w_1$ or $w_2$ (or a combination of the two frequencies).

- The vectors $\begin{bmatrix} 1 \\ r^{(1)} \end{bmatrix}$ and $\begin{bmatrix} 1 \\ r^{(2)} \end{bmatrix}$ are also independent of initial condition and only depend on the $K$ and $M$ matrices. These vectors are called the modes shapes of the system. For example, $\begin{bmatrix} 1 \\ r^{(1)} \end{bmatrix}$ is the mode shape corresponding to the natural frequency $w_1$. This mode shape, then shows the ‘form’ or ‘shape’ of the motion of
the system if the motion was comprised of \( w_1 \) term only. Indeed, there are initial conditions that would result in a motion that has a single frequency of \( w_1 \) where the ratio of the motion of \( x_1 \) to \( x_2 \) is \( r^{(1)} \) - i.e., the vector \( \begin{bmatrix} 1 \\ r^{(1)} \end{bmatrix} \) shows the ‘shape’ of the motion.

### 6.3 An important example (page 394 of the text)

Consider the system in the Figure 2, which is often called the ‘half car’ model (also see Figure 6.6.1 on page 395 of the text which is essentially the same!). Let

\[
x(t) : \text{Motion of center of gravity from the equilibrium position (positive is down)}
\]

\[
\theta = \text{rotation of the bar (positive direction is clockwise)}
\]

Figure 2: Active suspension model
The equation of motion can be written as

\[ m\ddot{x} = -(x + l_2 \sin \theta)k_2 - (x - l_1 \sin \theta)k_1 \]

\[ J\ddot{\theta} = -(x + l_2 \sin \theta)k_2 l_2 + (x - l_1 \sin \theta)k_1 l_1 \]

Now linearize the equations (i.e., \( \sin \theta = \theta, \cos \theta = 1, \ \theta \sin \theta = 0, \ etc \)) and put them in the matrix form, you will get

\[
\begin{bmatrix}
    m & 0 \\
    0 & J
\end{bmatrix}
\begin{bmatrix}
    \ddot{x} \\
    \ddot{\theta}
\end{bmatrix}
+ \begin{bmatrix}
    k_1 + k_2 & -(k_1 l_1 - k_2 l_2) \\
    -(k_1 l_1 - k_2 l_2) & k_1 l_1^2 + k_2 l_2^2
\end{bmatrix}
\begin{bmatrix}
    x \\
    \theta
\end{bmatrix}
= \begin{bmatrix}
    0 \\
    0
\end{bmatrix}
\tag{6.9}
\]

or in short hand

\[ M\ddot{x} + Kx = 0 \]

with obvious definitions for \( K, M, \) etc. Next, let us see what happens if we wanted to write the equations of motion in terms of \( y(t), \) the motion of a point \( e \) units to the left of the center of gravity? First, note that

\[ y = x - e \sin \theta \Rightarrow \dot{y} = \dot{x} - e \dot{\theta} \cos \theta \]

\[ \ddot{y} = \ddot{x} - e \ddot{\theta} \cos \theta + e\dot{\theta}^2 \sin \theta \]

which after linearization, becomes \( y = x - e \theta \) and \( \ddot{y} = \ddot{x} - e \ddot{\theta}, \) or

\[
\begin{bmatrix}
    y \\
    \theta
\end{bmatrix}
= \begin{bmatrix}
    1 & -e \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    x \\
    \theta
\end{bmatrix},
\begin{bmatrix}
    \ddot{y} \\
    \ddot{\theta}
\end{bmatrix}
= \begin{bmatrix}
    1 & -e \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    \ddot{x} \\
    \ddot{\theta}
\end{bmatrix}
\tag{6.10}
\]

Now we define

\[ T = \begin{bmatrix}
    1 & -e \\
    0 & 1
\end{bmatrix}^{-1} = \begin{bmatrix}
    1 & e \\
    0 & 1
\end{bmatrix} \tag{6.11}
\]

and using (6.10), we get

\[
\begin{bmatrix}
    \ddot{x} \\
    \ddot{\theta}
\end{bmatrix}
= \begin{bmatrix}
    1 & e \\
    0 & 1
\end{bmatrix}
\begin{bmatrix}
    \ddot{y} \\
    \ddot{\theta}
\end{bmatrix}
= T \begin{bmatrix}
    \ddot{y} \\
    \ddot{\theta}
\end{bmatrix}
\tag{6.12}
\]
Using this last expression in (6.9), the matrix form of the equations (in terms of \(y\) and \(\theta\)) becomes

\[
MT \begin{bmatrix} \ddot{y} \\ \ddot{\theta} \end{bmatrix} + KT \begin{bmatrix} y \\ \theta \end{bmatrix} = 0
\]

pre-multiplying by \(T^T\),

\[
T^TMT \begin{bmatrix} \ddot{y} \\ \ddot{\theta} \end{bmatrix} + T^TKT \begin{bmatrix} y \\ \theta \end{bmatrix} = 0
\]

where we can write in short hand as

\[
\hat{M} \begin{bmatrix} \ddot{y} \\ \ddot{\theta} \end{bmatrix} + \hat{K} \begin{bmatrix} y \\ \theta \end{bmatrix} = 0 \tag{6.13}
\]

where both \(\hat{M}\) and \(\hat{K}\) are symmetric, if the original \(K\) and \(M\) were symmetric and after some algebra

\[
\hat{M} = T^TMT = \begin{bmatrix} m & me \\ me & J \end{bmatrix}
\]

\[
\hat{K} = T^TKT = \begin{bmatrix} k_1 + k_2 & k_2(l_2 + e) - k_1(l_1 - e) \\ k_2(l_2 + e) - k_1(l_1 - e) & k_1(l_1 - e)^2 + k_2(l_2 + e)^2 \end{bmatrix}
\]

**Remark:** Note that instead of writing the equations of motion in terms of \(y\), we write the more natural form (i.e., in terms of \(x\)), and use the ‘transformation’ of (6.12) to relate the original coordinates to the desired coordinate. We have thus used a ‘coordinate transformation’.

What if someone asked you for the equations of motion in terms of the motion of the ends of the rod (let us call them \(x_1\) and \(x_2\))? We follow the procedure we used above in the change of coordinates.

\[
x_1 = x - l_1 \sin \theta, \quad x_2 = x + l_2 \sin \theta
\]

which after linearization, becomes

\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -l_1 \\ 1 & l_2 \end{bmatrix} \begin{bmatrix} x \\ \theta \end{bmatrix} \tag{6.14}
\]
Similarly, taking two derivatives of the $x_1$ equation above, for example, one gets

$$\ddot{x}_1 = \ddot{x} - l_1 \ddot{\theta} \cos \theta + l_1 \dot{\theta}^2 \sin \theta$$

Linearizing and combining with the $\ddot{x}_2$ equation, we get

$$\begin{cases} 
\ddot{x}_1 \\
\ddot{x}_2 
\end{cases} = \begin{bmatrix} 1 & -l_1 \\
1 & l_2 
\end{bmatrix} \begin{cases} \ddot{x} \\
\dot{\theta} 
\end{cases}$$

Now we define

$$\tilde{T} = \begin{bmatrix} 1 & -l_1 \\
1 & l_2 
\end{bmatrix}^{-1} = \frac{1}{l_1 + l_2} \begin{bmatrix} l_2 & l_1 \\
-1 & 1 
\end{bmatrix}$$  \hspace{1cm} (6.15)

as a result, we can write

$$\begin{cases} \ddot{x} \\
\dot{\theta} 
\end{cases} = \tilde{T} \begin{cases} \ddot{x}_1 \\
\ddot{x}_2 
\end{cases}$$  \hspace{1cm} (6.16)

with a similar form for the $x$ and $\theta$. Using this last expression in (6.9), and post multiplying by $\tilde{T}$ (exactly as the previous example) the equations of motion become

$$\tilde{T}^T \tilde{M} \tilde{T} \begin{cases} \ddot{x}_1 \\
\ddot{x}_2 
\end{cases} + \tilde{T}^T K \tilde{T} \begin{cases} x_1 \\
x_2 
\end{cases} = 0$$

or in compact form

$$\tilde{M} \begin{cases} \ddot{x}_1 \\
\ddot{x}_2 
\end{cases} + \tilde{K} \begin{cases} x_1 \\
x_2 
\end{cases} = 0$$

where, for example,

$$\tilde{M} = \tilde{T}^T \tilde{M} \tilde{T} = \frac{1}{(l_1 + l_2)^2} \begin{bmatrix} J + m_1 l_2^2 & m_1 l_2 - J \\
m_1 l_2 - J & m_1 l_2 - J + J 
\end{bmatrix}.$$  \hspace{1cm} (6.17)

Questions: Think about the following questions

• Is one set of coordinates ‘better’ than others?

• Are the natural frequencies one gets in the three different sets of $K$ and $M$ the same?

• Are the mode shapes one gets in the three different sets of $K$ and $M$ the same?

• Why did we pre-multiply by the ‘$T$’ matrix in each case?

• What happens if $k_1 l_1 = k_2 l_2$?
7 Forced Vibration and vibration Absorbers

Consider the system of Figure 3 (or Figure 5.13 of text on page 351). This is a relatively general two degrees of freedom system, in which some of the parameters are set equal to one another to simply the analysis.

Following regular procedure, the equation of motion are

\[ M \ddot{x} + Kx = F \]  \hspace{1cm} (7.17)

where

\[ M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \quad K = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}, \quad F = \begin{bmatrix} F_1 \cos wt \\ 0 \end{bmatrix} \]

where \( x_1(t) \) and \( x_2(t) \) are the absolute displacements of the masses and \( w \) is the frequency of the external forcing function. Parallel to the development of Chapter 3, we assume the solution has the following form

Figure 3: A simple two degree of freedom forced system
\[ \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} X_1 \cos wt \\ X_2 \cos wt \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \cos wt = \mathbf{X} \cos wt \quad (7.18) \]

Plugging this in the equation of motion, (7.17), we get

\[ [K - M w^2] \mathbf{X} = \begin{bmatrix} F_1 \\ 0 \end{bmatrix} \Rightarrow \mathbf{X} = [K - M w^2]^{-1} \begin{bmatrix} F_1 \\ 0 \end{bmatrix} \]

so that (if the inverse exists) the motion becomes

\[ \mathbf{x}(t) = [K - M w^2]^{-1} \begin{bmatrix} F_1 \\ 0 \end{bmatrix} \cos wt. \]

For 2 \times 2 systems, it is possible to carry out the inverse analytically. Noting that

\[ [K - M w^2]^{-1} = \begin{bmatrix} 2k - mw^2 & -k \\ -k & 2k - mw^2 \end{bmatrix}^{-1} = \frac{1}{(3k - mw^2)(k - mw^2)} \begin{bmatrix} 2k - mw^2 & k \\ k & 2k - mw^2 \end{bmatrix} \]

we get

\[ \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \frac{F_1(2k-mw^2)}{(3k-mw^2)(k-mw^2)} \\ \frac{F_1k}{(3k-mw^2)(k-mw^2)} \end{bmatrix} \cos wt \quad (7.19) \]

The (normalized) magnitude of each \( x_i(t) \) as \( w \) changes similar to the plots shown on page 466 of the text. Note some very interesting properties:

- As \( w^2 \) gets closer to either \( \frac{k}{m} \) or \( \frac{3k}{m} \) the amplitude of both masses gets very large; i.e, we observe the ‘resonance’ behavior. Recall that for this problem, the natural frequencies are exactly \( \sqrt{\frac{k}{m}} \) and \( \sqrt{\frac{3k}{m}} \)

- We have ignored damping because (as you might have expected from the one degree of freedom case!) it makes the problem much much more complicated - without adding new insight. The peaks will be finite and valleys will be shallower of course. We will deal damping in some detail in Chapter 7.
• In the motion of $x_1(t)$, see what happens as $w^2$ gets close to $\frac{2k}{m}$? Note that while $x_2(t) \neq 0$, we will get $x_1(t) = 0!!$ This value also depends on mass and stiffness properties of the system. If this condition holds, the system behaves as a ‘vibration absorber’. That is, the second mass ‘absorbs’ the vibration that the forcing function might create and allows the first mass to stay stationary. We will study this in more detail in Chapter 7.

• In the language of controls (ME170B), $\frac{2k}{m}$ is the zero for the transfer function from $F$ to $x_1$ (and the natural frequencies are poles)
8 Multi-degrees-of-freedom Systems

8.1 Background

First, we start by writing the equations of motion

\[ \ddot{x}(t) + Kx(t) = F(t) \quad \text{with no damping} \] (8.1)

\[ \ddot{x}(t) + C\dot{x}(t) + Kx(t) = F(t) \quad \text{with damping} \] (8.2)

To do this, three different approaches can be used:

a. Directly from Newton’s laws

b. By inspection

c. From the energy expressions. For this approach, first note that

\[ T = \text{total kinetic energy} = \frac{1}{2} \dot{x}^T M \dot{x} \]

\[ U = \text{total potential energy} = \frac{1}{2} x^T K x. \]

If you can shape the energy expression into the form on the right hand side, you have obtained the \( M \) and \( K \) matrices. From now on, we will assume (without any loss of generality) that both \( M \) and \( K \) are symmetric.

Next, let us analyze the fundamental properties of the system. We will consider the undamped and free vibration, for the moment.

8.2 Undamped, free vibration

\[ \ddot{x}(t) + Kx(t) = 0 \] (8.3)
Similar to what we did for the one and two degrees of freedom systems, we search for motions of form \( x(t) = X \cos \omega t \), where \( x(t) \) is the vector of the coordinates we have used (for example, displacements of each degree of freedom). We get

\[
[-w^2 M + k]X = 0
\]  

(8.4)

where non-trivial solution can be found ONLY IF \( \det [-w^2 M + k] = 0 \). So we set this determinant to zero and find the values of \( w \) that result in this determinant becoming zero; i.e.,

\[
\det [-w^2 M + K] = 0 \implies w_1, w_2 \cdots w_n
\]

where these \( w_i \)'s are called the natural frequencies (you will soon see why!) and are roots of the determinant equation. It is relatively easy to show that this equation has \( n \) solutions for \( w^2 \). It is somewhat more difficult to show that all solutions are nonnegative (i.e., either zero or positive numbers). For each \( w_i \), solve the following equation for \( X_i \)

\[
[-w_i^2 M + K]X_i = 0.
\]

As before, if \( X_i \) satisfies this equation, so does \( aX_i \), for any scalar \( a \). The pairs \((w_i, X_i)\) are called the \( i^{th} \) natural frequency and mode shape of the system. Note the similarity between what we did here and the development of the two degree of freedom systems. The problem is that for \( n \geq 3 \), this process is not practical. Let us start by rewriting (8.4),

\[
[-w^2 M + k]X = 0 \implies KX = w^2 MX \quad \text{or} \quad M^{-1} KX = w^2 X.
\]  

(8.5)

The last 2 equations are two forms of the well known ‘eigenvalue’ problem, for which a great deal of reliable, efficient and accessible software has been developed; i.e., in practice we follow

\[
(8.5) + \text{ computer programs } \implies n \text{ pairs of } (w_i^2, X_i)
\]

8–2
where \( w_i^2, X_i \) are called the \( i^{th} \) eigenvalue and its corresponding eigenvector for (8.5), respectively.

The following properties of the eigenvalue problem are well known, and can be shown through basic linear algebra:

- All \( n \) natural frequencies or eigenvalues of (8.5); i.e., \( w_1^2, w_2^2, \ldots, w_n^2 \), are real and non-negative.
- All \( n \) mode-shapes or eigenvectors of (8.5), \( X_1, X_2, \ldots, X_n \), corresponding to respective \( w_i \)'s, are real and are linearly independent of one another.

Why are we interested in these? For the following reasons

**a.** Physical understanding of the motion. Think of the vibrating string, for analogy. The \( w_i \)'s are similar to the ‘harmonics’ of the string, while the mode shapes are similar to the shape corresponding to each harmonic (the half sine-waves in the vibrating string). They give important information about resonance (more on this below) and the points where each mode shape has maximum or minimum motion (for sensor/actuator placement, attachment design, etc).

**b.** Damping analysis, uncoupling of the equation of motions, and order reduction: These three are combined because they all follow from the orthogonality property of mode shapes (or eigenvectors), see Section 6.10.2 of the text for details.

\[
X_i^T M X_j = 0 \quad \text{if} \quad i \neq j, \quad \text{and} \quad X_i^T M X_i \triangleq m_i \quad (8.6)
\]

\[
X_i^T K X_j = 0 \quad \text{if} \quad i \neq j, \quad \text{and} \quad X_i^T K X_i \triangleq k_i. \quad (8.7)
\]

These \( m_i \)'s and \( k_i \)'s are sometimes called generalized mass and stiffness of the \( i^{th} \) coordinate. Note that since we can have any value between 1 and \( n \) for \( i \) or \( j \), equations (8.6) and
(8.7) represent \( n^2 \) equations, each. As an (easy) exercise, convince yourself that putting all \( n^2 \) equations in a matrix form results in the following

\[
[ X_1 \cdots X_n ]^T M [ X_1 \cdots X_n ] = B^T M B = \text{diag}[m_i]
\]

(8.8)

\[
[ X_1 \cdots X_n ]^T K [ X_1 \cdots X_n ] = B^T M B = \text{diag}[k_i]
\]

(8.9)

where we have used \( B \) as a short hand for the matrix of mode shapes; i.e.,

\[
B = [ X_1 \ X_2 \cdots X_n ].
\]

(8.10)

In (8.8) and (8.9), \( \text{diag}[m_i] \) mean a diagonal matrix with \( m_i \) as its \((i, i)\) entry. Finally, let us normalize the mode shapes according to

\[
\hat{X}_i = \frac{1}{\sqrt{m_i}} X_i.
\]

(8.11)

Again, it is an easy exercise to show that

\[
[ \hat{X}_1 \cdots \hat{X}_n ]^T M [ \hat{X}_1 \cdots \hat{X}_n ] = \hat{B}^T M \hat{B} = I
\]

(8.12)

while with little work, one can show

\[
[ \hat{X}_1 \cdots \hat{X}_n ]^T K [ \hat{X}_1 \cdots \hat{X}_n ] = \hat{B}^T K \hat{B} = \text{diag}[k_i] = \text{diag}[w_i^2]
\]

(8.13)

where \( w_i^2 \)'s are the natural frequencies! We have used \( \hat{B} \) as a short hand for the matrix of normalized mode shapes; i.e.,

\[
\hat{B} = [ \hat{X}_1 \ \hat{X}_2 \cdots \hat{X}_n ].
\]

(8.14)

EXERCISE: Verify the second equality in (8.13).

SKETCH: Recall that for matrices \( A, B, C \), we have \( \det(ABC) = \det A \times \det B \times \det C \). Now by standard results from the eigenvalue problem, \( \det \hat{B} \neq 0 \). Set \( \det \hat{B}^T [-w^2 M + K] \hat{B} = 0 \)

8–4
8.3 Forced Vibration

We go back to equation (8.1),

\[ M\ddot{x}(t) + Kx(t) = F(t) = F_0 \cos wt \quad (or \ cos wt) \quad (8.15) \]

From math, ME170 or early chapters of ME147, we expect that the steady state response (i.e., particular solution) be of the form \( x(t) = X \cos wt \). Here, \( w \) is the frequency of the forcing function and is known (as compared to free vibration) and \( X \) is NOT the mode shape, and simply is the amplitude of response. Substitute this solution in (8.15)

\[ \left[ -w^2 M + K \right] X = F_0 \implies X = \left[ -w^2 M + K \right]^{-1} F_0 \]

ONLY IF THE INVERSE EXISTS, that is only if the determinant of the matrix is not zero. But we know that at \( w = w_i \) (i.e., any of the natural frequencies discussed above) this determinant \textit{is} zero. Therefore, these \( w_i \)'s are frequencies that if the system is forced at, problems can arise, i.e., at the very least we cannot use what we have learned above. Further below, we will see that the response goes to infinity! (hence resonance and the name ‘natural frequencies’). To clarify this, we will use the mode shapes.

Next, let us define the following coordinates

\[ x(t) = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix} \quad q(t) = Bq(t). \quad (8.16) \]

Remember that \( x(t) \) was a vector of physically meaningful coordinates (e.g., displacement, rotation or velocity of specific points of the system). Vector \( q(t) \), however, is vector of coordinates \( q_i(t) \), which in general are not physically meaningful. The best we can say is that each \( q_i(t) \) represents how much of the overall motion, at time \( t \), is due to the mode shape (and natural frequency) \( i \).
Use (8.16) in (8.15) and premultiply (8.15) by $B^T$, and using the identities in (8.8)-(8.10), you will get

$$\text{diag}[m_i] \ddot{q}(t) + \text{diag}[k_i] q(t) = B^T F_o \cos wt, \quad (\text{or } \sin)$$

(8.17)

i.e., $n$ un-coupled equations of motion, which can be solved a lot easier (even by hand!). Let us try to solve the problem, as usual we assume $q(t) = Q \cos wt$ and, after some simplification, (make sure you see how the following equation is obtained)

$$(-w^2 m_i + k_i) Q_i = X_i^T F_o \implies Q_i = \frac{X_i^T F_o}{(-w^2 m_i + k_i)}$$

(8.18)

Note that, according to (8.18), if $w^2$ gets close to $\frac{k_i}{m_i}$ for any $i$, the corresponding $q_i(t)$ goes to infinity and, due to (8.16), so does $x(t)$ (i.e., resonance!) The behavior of each $q_i(t)$, with respect to the forcing frequency $w$ is similar to the plots in Figures 3.3 and 3.11 of the book. From (8.13), one can see that these resonance values are the same as the natural frequencies of the system.

Lastly, if forcing frequency, $w$, is not at resonance values, (8.18) is used to calculate $q(t)$. From (8.16) then, we can find $x(t)$. Notice that if $w$ is close to any of the $w_i$’s, then the corresponding $Q_i$ in (8.18) will be large. As a result if the forcing frequency is relatively close to one of the modes, then that mode will have strong presence in the overall response (see (8.16)). Sometimes, we use the following phrase: if $w$ is close to $w_i$, then the $i^{th}$ mode will be highly excited.

### 8.4 Damped Vibration

In this case, we have the following for the equations of motion

$$M \ddot{x}(t) + C \dot{x}(t) + K x(t) = F(t)$$

(8.19)
In general, little is known about the true nature of the damping in many important applications. If $C$ matrix in (8.19) does not have any specific structure, the resulting response can be quite complicated and hard to understand. Suppose however, that the following is true for some positive scalars $\alpha$ and $\beta$

$$C = \alpha M + \beta K$$  \hspace{1cm} (8.20)

Using (8.16) and (8.20) in (8.19), and premultiplying (8.19) by $B^T$, similar to (8.17) yields,

$$\text{diag}[m_i] \ddot{q}(t) + \text{diag}[\alpha m_i + \beta k_i] \dot{q}(t) + \text{diag}[k_i] q(t) = B^T F_o \cos \omega t$$  \hspace{1cm} (8.21)

which is $n$ decoupled second order differential equations, each of the type

$$m \dddot{z} + c \dot{z} + k z = f.$$  

The behavior of each $q_i(t)$, with respect to the forcing frequency, is thus the same as Figure 3.11 of the book. As a result, effective damping, exponential decay and other similar properties can be estimated or approximated for the system. Note that this simple transformation of the original coupled equations into $n$ decoupled equations can be achieved because the damping matrix is of the form in (8.20).

There are other variations of this method, including variations of (8.20), but the development above captures the crucial aspects of it. In many cases, $\alpha$ and $\beta$ are chosen for the best fit in data collected in lab.

### 8.5 Modal Truncation

Think about a large order model (e.g., that of a wing of an aircraft, satellite or a tall building). Say, 50 degrees of freedom. After some testing, it may be determined that the loading is from very low frequencies, and thus only the first three (again as example) modes
are excited. Since the other 47 modes are not excited, ignoring them is not a bad idea (as a
first approximation). This can be done through the following: Start with the full equations
of motion \((n = 50)\)

\[
M \ddot{x}(t) + Kx(t) = F.
\]

Using (8.16), we write

\[
x(t) = Bq(t) = X_1q_1(t) + \cdots X_{50}q_{50}(t).
\]

The truncated system is based on the first three modes only; i.e.,

\[
x_{tr}(t) = X_1q_1(t) + X_2q_2(t) + X_3q_3(t) = B_{tr}q_{tr}(t)
\]

where \(B_{tr}\) is the first three columns of \(B\) (a 50 \(\times\) 3 matrix) and \(q_{tr} = [q_1\ q_2\ q_3]^T\). The
reduced (or truncated) model will then be the following 3 \(\times\) 3 matrix equation

\[
B_{tr}^TMB_{tr} \ddot{q}_{tr}(t) + B_{tr}^TCB_{tr} \dot{q}_{tr}(t) + B_{tr}^TKB_{tr} q_{tr}(t) = B_{tr}^TF.
\]