6 OUTPUT FEEDBACK DESIGN

When the whole state vector is not available for feedback, i.e., we can measure only
\[ y = Cx. \]

6.1 Review of observer design

Recall from the first class in linear systems that a simple control law would be
\[ u = Kx \Rightarrow \dot{x} = (A + BK)x \]
where \( K \) is chosen so that \( A + BK \) is stable (from pole placement of LQR, etc).

Now if you cannot measure \( x \), then you use an output feedback design. Static output feedback design; i.e., \( u = Ky \) turns out to be relatively hard to solve (unless you do trial and error) - more on this later. The most common - and systematic approach is to use a dynamic output feedback, where the controller (or compensator) has its own dynamics (recall the typical compensator box from classical control course). The simplest form is an observer structure; i.e., use \( u = K\hat{x} \) where \( \hat{x} \) is an estimate for the actual \( x \) and comes from a copy of the model we construct with our control hardware (or software)

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
\dot{\hat{x}} &= A\hat{x} + Bu - L(y - C\hat{x}) \\
u &= K\hat{x}
\end{align*}
\]

(6.1)

(6.2)

(6.3)

The trick, in this simple approach, is to pick a good \( L \) such that \( \hat{x} \rightarrow x \) relatively soon. First, let us write the model in terms of \( x \) and \( e = x - \hat{x} \)

\[
\begin{align*}
\dot{x} &= Ax + BK\hat{x} = (A + BK)x - BKe \\
\dot{e} &= (A + LC)e 
\end{align*}
\]

(6.4)

(6.5)

(6.6)

which can be written as the following for \( x_{cl}^T = (x^T \quad e^T) \)

\[
\dot{x}_{cl} = A_{cl}x_{cl} \, , \, A_{cl} = \begin{bmatrix} A + BK & -BK \\ 0 & A + LC \end{bmatrix}
\]

since \( det(A_{cl}) = det(A + BK) \cdot det(A + LC) \), we have closed loop stability (i.e. \( e(t) \rightarrow 0 \) as time gets larger, which means \( \hat{x} \rightarrow x \)) as long as \( A + LC \) stable. Note that this is independent of the choice of \( K \) - a trivial case of ‘separation principle,’ Also note that the poles of \( A + LC \) set how fast \( e(t) \) dies. It is common to use a role of thumb that days the least stable pole of \( A + LC \) should be three times as fast as the dominant modes of \( A + BK \).
6.2 The Kalman filter and the LQG

In this subsection, we will review very briefly Kalman filter equations for the Linear Quadratic Gaussian problem (LQG). Due to the time limitation, this review will be extremely brief.

Consider the following stochastic linear system

\[
\begin{align*}
\dot{x} &= Ax + Bu + \Gamma \zeta(t) \\
y &= Cx + \eta(t)
\end{align*}
\]

(6.7)

where \(\zeta(t)\) and \(\eta(t)\) are vector random processes (i.e., the process noise and measurement noise, respectively).

After ignoring a great deal of effort (and potential pitfalls) with respect to the well-posedness of (6.7), we assign

\[
\begin{align*}
\mathbb{E}[\zeta(t)] &= \mathbb{E}[\eta(t)] = 0 \quad \text{(zero mean)} \\
\mathbb{E}[\zeta(t)\eta^T(\tau)] &= 0 \quad \text{(uncorrelated)} \\
\mathbb{E}[\zeta(t)\zeta^T(\tau)] &= Q_o \delta(t-\tau), \quad Q_o \geq 0 \\
\mathbb{E}[\eta(t)\eta^T(\tau)] &= R_o \delta(t-\tau), \quad R_o > 0
\end{align*}
\]

(6.8)

where \(\mathbb{E}\) is the expectation operator (think ensemble average, or in the case of ergodic signals, time averages). Relying only on the measurement \(y(t)\), we wish to ‘estimate’ the \(x(t)\) - and denote the estimate by \(\hat{x}\) - such that the following error is minimized:

\[
\begin{align*}
\{ e(t) &= x(t) - \hat{x}(t) \\
\text{minimize} \quad \mathbb{E}[e^T(t)e(t)] &= \min \mathbb{E}[\|x(t) - \hat{x}(t)\|^2].
\end{align*}
\]

(6.9)

After a great deal of work (roughly two quarters of stochastic processes worth!) the following is obtained for the steady state case:

Observer Equation:

\[
\begin{align*}
\dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + K_f[y(t) - C\hat{x}(t)] \\
K_f &= SC^TR_o^{-1}
\end{align*}
\]

(6.10, 6.11)

\[
AS + SA^T - SC^TR_o^{-1}CS + \Gamma Q_o \Gamma^T = 0.
\]

(6.12)

Note that with this observer, the error equation becomes

\[
\dot{e}(t) = [A - K_f C]e(t) + [\Gamma - K_f]\begin{bmatrix} \zeta(t) \\ \eta(t) \end{bmatrix}.
\]

(6.13)

After noting the similarity of these equations to those of the LQR method, we have the following:
Remark 6.1. The Riccati equation in (6.12) resembles the one encountered in LQR. Notice the duality between the two, by replacing \( B \) with \( C^T \) and \( A \) by \( A^T \). As a result, a great many of the results and techniques we discussed earlier apply here, as well. For example: if \( (A,C) \) is observable and \( (A,\Gamma) \) is controllable, then (6.12) has a unique solution, \( S > 0 \), and \( (A-K_fC) \) is stable.

Remark 6.2. In (6.10) if the noise terms are ignored, then we have an observer which is obtained from (6.11)-(6.12), and results in a stable closed loop. Further, if the model has noise (as in (6.7)) then this observer minimizes (6.9), as well. This is the approach we will choose; i.e., we will use (6.12) and (6.11) to design 'desirable' observers (rather than pole placement methods, for example). Lastly, matrices \( Q_o \) and \( R_o \) can be interpreted as the intensity of the process noise and measurement noise, respectively. A very large \( R_o \), for example, denotes high levels of measurement noise. One might expect that such a system would end up with small observer gains (why?), which is indeed true. What is the dual problem in LQR?

6.3 The Linear Quadratic Guassian Compensator - LQG

Consider the dynamical system in (6.7), subject to (6.8). The LQG problem is to design a controller of the form

\[
    u(t) = f[y(\tau), \tau \leq t]
\]

(6.14)

to minimize

\[
    J = \mathcal{E}[\lim_{T \to \infty} \frac{1}{T} \int_0^T (x^T H^T H x + u^T R u) \, dt]
\]

(6.15)

under the following assumptions

\[
\begin{align*}
    (A,B), & \text{ and } (A\Gamma) \text{ controllable(stabilizable)} \\
    (A,C) \text{ and } (A,H) \text{ observable} \\
    R_o > 0, & \text{, } R > 0 \text{, } Q_o \geq 0.
\end{align*}
\]

(6.16)

The solution to this problem is the following:

\[
\begin{align*}
    u(t) &= -K_c \dot{x}(t) \\
    K_c &= R^{-1} B^T P \\
    PA + A^T P - PBR^{-1} B^T P + H^T H &= 0 \\
    \dot{x}(t) &= A\dot{x}(t) + Bu(t) + K_f[y(t) - C\dot{x}(t)] \\
    K_f &= SC^T R_o^{-1} \\
    AS + SA^T - SC^T R_o^{-1} CS + \Gamma Q_o \Gamma^T &= 0.
\end{align*}
\]

(6.17)
Remark 6.3. Equation (6.17) implies that the optimal solution can be separated into full state controller and observer design. This principle of separation in stochastic control works similar to the one encountered in pole-placement type controllers (in deterministic setting). Its proof, however, is quite complicated. Note the control consists of a LQR step plus an observer step (which is the steady state Kalman filter).
6.4 Problem Set

Exercise 6.4. Ignoring the noise (i.e., $\eta$ and $\zeta$), write the combined closed-loop state space form (in terms of state variables $x$ and $\hat{x}$).

Exercise 6.5. Define the error to be $x - \hat{x}$. Write the combined closed-loop state state form in terms of state variables $x$ and $e$. Are the eigenvalues of the closed loop system the same as in the previous exercise? Why?

Exercise 6.6. Again, set the noise to be zero. What is the transfer function of the compensator? That is, if we write the control as in $u(s) = H(s)y(s)$, what is $H(s)$, where ‘$x(s)$’ denotes the Laplace Transform of $x(t)$. Try to draw the block diagram of this problem.

Exercise 6.7. Is the compensator (i.e., $H(s)$) stable? What are some of the possible problems with unstable compensators?