Polar Coordinates and Rotating Frames

Consider the position of the point mass \( A \), which is moving around the fixed point \( O \). The position vector for \( A \) can be written in terms of Cartesian coordinates as

\[
\mathbf{r} = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j}
\]

We will now assume the following

- The counter clockwise rotation is considered in positive \( k \) direction (recall the old ‘right hand rule’)
- Both \( r \) (the length) and \( \theta \) (the orientation) can change with time and, thus have time derivatives
- We are seeking expressions for absolute velocity and absolute acceleration of the point mass \( A \) (i.e., we are seeking \( \dot{\mathbf{r}} = \mathbf{v} \) and \( \ddot{\mathbf{r}} = \mathbf{a} \))

1 Method I: Fixed Coordinates

In this section, we try to keep everything in the x-y (or \( i - j \)) coordinates. As you will see, it is conceptually simple, but becomes rather messy and confusing. We start by taking the time derivative of the position vector \( \mathbf{r} \). We will use the following notations:

- \( \frac{d}{dt} r = \dot{r} \) (i.e., change in length, or speed)
- \( \frac{d}{dt} \dot{r} = \ddot{r} \) (i.e., change in speed or acceleration)
• \( \frac{d}{dt} \theta = \dot{\theta} \) (i.e., angular velocity)

• \( \frac{d}{dt} \sin \theta = \cos \theta \dot{\theta} \) (recall the chain rule)

Now, we can go ahead and take derivative of the position vector in (1).

\[
v = \frac{d}{dt} \mathbf{r} = (\dot{r} \cos \theta - r \dot{\theta} \sin \theta) \mathbf{i} + (\dot{r} \sin \theta + r \dot{\theta} \cos \theta) \mathbf{j}
\]

(2)

and for acceleration

\[
a = \frac{d}{dt} \frac{d}{dt} \mathbf{r} = (\ddot{r} \cos \theta - r \dot{\theta}^2 \sin \theta - r \dot{\theta} \dot{\theta} \sin \theta - r \dot{\theta} \dot{\theta} \cos \theta) \mathbf{i} +
\]

\[
(\dot{r} \sin \theta + r \dot{\theta} \cos \theta + \dot{r} \dot{\theta} \cos \theta + r \ddot{\theta} \cos \theta - r \dot{\theta}^2 \sin \theta) \mathbf{j}
\]

(3)

2 Method II: Polar Coordinates

Let us use the polar coordinates as define by:

• \( \mathbf{e}_r \) a unit vector, radially out from O to A

• \( \mathbf{e}_\theta \) a unit vector, perpendicular to \( \mathbf{e}_r \) such that \( \mathbf{k} \) (out of plane of the paper) is the third coordinate (with the right hand rule).

• From the right hand rule, we have \( \mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{k} \), \( \mathbf{e}_\theta \times \mathbf{k} = -\mathbf{e}_r \)

Note that both \( \mathbf{e}_r \) and \( \mathbf{e}_\theta \) rotate and change their orientation (direction). Therefore, they have time derivatives, though their lengths are constants. Using the polar coordinates, we can write the position vector as

\[
\mathbf{r} = r \mathbf{e}_r
\]

(4)

Taking derivative of the position vector in (4) yields

\[
v = \frac{d}{dt} \mathbf{r} = \dot{r} \mathbf{e}_r + r \dot{\mathbf{e}}_r
\]

(5)

BUT what is this \( \dot{\mathbf{e}}_r \)? It is the time derivative of a rotating vector. To calculate it, we go back to Figure 1 and note that

\[
\mathbf{e}_r = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}
\]

(6)

\[
\mathbf{e}_\theta = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j}
\]

(7)

in which \( \mathbf{i} \) and \( \mathbf{j} \) are constant (i.e., do not change with respect to time) but \( \theta \) has a time derivative (\( \dot{\theta} \)). Therefore we have

\[
\dot{\mathbf{e}}_r = -\dot{\theta} \sin \theta \mathbf{i} + \dot{\theta} \cos \theta \mathbf{j} = \dot{\theta} (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) = \dot{\theta} \mathbf{e}_\theta
\]
similarly
\[ \dot{\epsilon}_\theta = -\dot{\theta} \cos \theta \hat{\imath} - \dot{\theta} \sin \theta \hat{j} = -\dot{\theta} (\cos \theta \hat{\imath} + \sin \theta \hat{j}) = -\dot{\theta} \epsilon_r \]
that is
\[ \dot{\epsilon}_r = \dot{\theta} \epsilon_\theta, \quad \dot{\epsilon}_\theta = -\dot{\theta} \epsilon_r. \tag{8} \]

Equation (8) is pretty important and you will end up memorizing it in all likelihood. It is a special case for a well known equation in kinematics which is quite useful. Here we go (!)

*If \( \mathbf{n} \) is a rotating unit vector, which is rotating with angular velocity of \( \Omega \), then its derivative is calculated from*
\[ \frac{d}{dt} \mathbf{n} = \Omega \times \mathbf{n} \tag{9} \]

Note that in (9), \( \Omega \) is the cause of the rotation for the unit vector \( \mathbf{n} \). Now, going back to our previous discussion, recall that \( \epsilon_r \) and \( \epsilon_\theta \) are rotating unit vectors that are rotating with angular velocity of \( \dot{\theta} \) in positive \( \hat{k} \) direction (if the rotation in clockwise, simply put a negative sign in front of \( \dot{\theta} \)). As a result, (9) gives
\[ \dot{\epsilon}_r = \dot{\theta} \hat{k} \times \epsilon_r = \dot{\theta} \epsilon_\theta \tag{10} \]
\[ \dot{\epsilon}_\theta = \dot{\theta} \hat{k} \times \epsilon_\theta = -\dot{\theta} \epsilon_r \tag{11} \]
which is the same as (8) above!!

Now, let us go back to our velocity term \( \mathbf{v} \): from (5) and (8) or (10)
\[ \dot{\mathbf{r}} = \mathbf{v} = \dot{r} \epsilon_r + r \dot{\theta} \epsilon_\theta \tag{12} \]

The question is: are (2) and (12) the same? (they do not look like it!). To see this, simply use the expression for \( \epsilon_r \) and \( \epsilon_\theta \) (from (6) and (7)) in (12). We thus get
\[ \dot{r} \epsilon_r + r \dot{\theta} \epsilon_\theta = \dot{r} (\cos \theta \hat{\imath} + \sin \theta \hat{j}) + r \dot{\theta} (-\sin \theta \hat{\imath} + \cos \theta \hat{j}) = (\dot{r} \cos \theta - r \dot{\theta} \sin \theta) \hat{\imath} + (\dot{r} \sin \theta + r \dot{\theta} \cos \theta) \hat{j} \]
which is exactly the right hand side of (2)!! Finally, let us do the acceleration by taking the derivative of \( \mathbf{v} \) in (12) which yields
\[ \mathbf{a} = \ddot{\mathbf{r}} = \ddot{r} \epsilon_r + \dot{r} \dot{\epsilon}_r + \dot{r} \dot{\theta} \epsilon_\theta + r \ddot{\theta} \epsilon_\theta + r \dot{\theta} \dot{\epsilon}_\theta \]
\[ = \ddot{r} \epsilon_r + \dot{r} \dot{\theta} \epsilon_\theta + \dot{r} \dot{\theta} \epsilon_\theta + r \ddot{\theta} \epsilon_\theta - r \dot{\theta}^2 \epsilon_r \]
which can be written as
\[ \mathbf{a} = \ddot{\mathbf{r}} = (\ddot{r} - r \dot{\theta}^2) \epsilon_r + (r \ddot{\theta} + 2 \dot{r} \dot{\theta}) \epsilon_\theta \tag{13} \]
Believe or not, this is exactly the same as the right hand side of (3) after using the expressions for $e_r$ and $e_\theta$ as we did for velocity. This last detail is left as an exercise, however! The textbook comes up with the same equations, albeit with a slightly different approach (see Section 11.14 and equations 11.43 and 11.44).

The two expressions (using the two coordinate systems) are equivalent. In some problems the polar coordinates (which are conceptually hard but easy to manipulate) are much more desirable that the Cartesian coordinates (which are easy to understand but quite messy). Typically, when there are major rotations or central forces, the polar coordinates are quite useful (examples: robotics or orbital mechanics).

2.1 Special Case: constant $r$

In your freshman physics, you had equations that were somewhat similar. In fact, most of you have seen the special case of these equations for the case constant $r$ (spinning a mass with a rope, etc.) When $r$ is constants, $\dot{r}$ and $\ddot{r}$ are both zero. As a result, we have $v = r \dot{\theta} e_\theta$. You may remember this as velocity being perpendicular to the radius. Note that ‘speed’ (i.e., magnitude of velocity vector) is now simply $r|\dot{\theta}|$.

For acceleration, note that we will have two component: in $e_r$ direction (which is radially out) we have $-r \ddot{\theta} e_r$ which can also be written as $-\frac{r^2 \dot{\theta}^2}{r} e_r$ or $\frac{\text{speed}^2}{r}$ radially inward! The $e_\theta$ component was $r \ddot{\theta}$ which is simple $\frac{d}{dt} \text{speed}$! So the equations you may remember from physics are simply special cases here.

YOU NEED TO BE CAREFUL: these simplified equations are only useful when $r$ is constant. One of the most common mistakes in this class is using these equations when $r$ is not constant.