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DUAL QUATERNION SYNTHESIS OF A PARALLEL 2-TPR ROBOT

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ABSTRACT

This paper presents a synthesis methodology for parallel robots based on the dual quaternion synthesis of serial constrained robots, that is, serial robots with less than six degrees of freedom. This methodology uses a dual quaternion formulation of the kinematics equations. The goal of the synthesis problem is to determine the dimensions of the robot from a specification of its workspace. The workspace of a constrained serial robot is a subset of the group of spatial transformations which can, in turn, be represented by a subset of dual quaternions. The basic approach is to specify the dual quaternion kinematics equations for each transformation of a discrete approximation to the desired workspace. The structure of these dual quaternions allows a systematic elimination of joint parameters for many constrained serial robot topologies. While the actual finite position synthesis methods are suited only to a small set of simple spatial serial chains, this new formulation can be applied to many serial chains with a greater number of degrees of freedom and joints. The multiple solutions obtained can in turn be combined to create parallel robots. Here we present the theory and formulate and solve the synthesis equations for the 2 TPR parallel robot.

1 Introduction

Our approach to the exact finite position synthesis of parallel robots is based on combining the multiple solutions of the synthesis of serial chains. The combination of the serial chains introduces constraints on the movement of the individual chains that can affect the smooth movement of the platform. To avoid this, the designer must impose a series of conditions in the set of solutions of the serial synthesis problem. The set of constraints and conditions is usually solved by using optimization. Chedmail 1998 and Gossellin 1998 present optimization techniques for design serial and parallel robotic system, respectively, that provide desired properties of the workspace. Murray 2000 presents a similar methodology applied to planar platforms.

The goal of the geometric design of a serial chain is to compute its kinematic parameters from a specification of the workspace. In this paper, we develop a new methodology for the geometric design of parallel robots composed of serial chains that have less than six degrees of freedom. The proposed synthesis methodology is an extension of the kinematic synthesis of linkages (McCarthy 2000a), which is based on finding the geometric constraints of the serial chain. The previous methodology is developed in a case by case basis and has been solved only for a few spatial cases. The advantage of the new methodology, based on the expression of the kinematic equations in dual quaternion formulation, is that it can be applied to serial chains with up to five degrees of freedom and joint axes. The multiple solutions obtained with this method can in many cases be combined to create parallel robots. The range of topologies of parallel robots that can be created using the dual quaternion synthesis is broader than by using the conventional synthesis methodology.

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1.1 Linkage Synthesis

Spatial linkage synthesis uses the geometric properties of a serial chain to formulate algebraic equations that must be satisfied at each of a discrete set of positions in the workspace (Suh and Radcliffe 1979). This yields algebraic equations that are solved to determine the dimensions of the chain. Also see McCarthy (2000b). Examples of this are the synthesis of spatial RR chains (Suh 1969, Tsai and Roth 1973, Perez and McCarthy 2000), spatial CC chains (Chen and Roth 1969, Huang and Chang 2000) and SS chains (Innocenti 1994, Liao and McCarthy 2000).

Recently, Mavroidis and Lee used the kinematics equations of the spatial RR and RRR robots to formulate its design equations. This approach introduces the joint parameters of the chain at each of the goal positions as additional variables in the design equations (Mavroidis and Lee 2001, Lee and Mavroidis 2002). The advantage is that it can be systematically applied to a broad range of robotic systems. Larochelle 2000 uses planar quaternion optimization for the approximate synthesis of planar chains.

1.2 Overview

In this paper, we follow Mavrodis' basic ideas, however, we use successive screw displacements (Gupta 1986, Tsai 1999) formulated in terms of dual quaternions to represent the kinematics equations of the robot. Dual quaternions were introduced to linkage analysis by Freudenstein and Yang (1964). They form an eight dimensional Clifford algebra that contains a subset, known as unit dual quaternions, which is isomorphic the group of spatial displacements (McCarthy 1990). Also see Angeles 1998.

There are two advantages in this formulation. The first is that successive screw displacements provide a convenient formulation for the kinematics equations in terms of the joint axes directly. Secondly, it reduces the number of equations obtained in each goal position from 12 to 8.

2 The Kinematics Equations

The kinematics equations of the robot equate the 4×4 homogeneous transformation $[D]$ between the end-effector and base frame. to the sequence of local coordinate transformations along the chain (Craig 1989),

$$\begin{aligned}[D] = [G][Z(\theta_1, d_1)][X(\alpha_{12}, a_{12})][Z(\theta_2, d_2)]\ldots \\ [X(\alpha_{n-1,n}, a_{n-1,n})][Z(\theta_n, d_n)][H].\end{aligned} \tag{1}$$

The parameters (θ, d) define the movement at each joint and (α, a) are the length and twist of each link, collectively know as the Denavit-Hartenberg parameters. The transformation $[G]$ defines the position of the base of the chain relative to the fixed frame, and $[H]$ locates the tool relative to the last link frame.

2.1 Successive Screw Displacements

These kinematics equations can be transformed into successive screw displacements choosing a reference position $[D_0]$. We then compute $[D_{0i}] = [D_i][D_0]^{-1}$, that is

$$\begin{aligned}&[D_{0i}] = [D_i][D_0]^{-1} = \\ &([G][Z(\theta_{1i}, d_{1i})]\ldots[Z(\theta_{ni}, d_{ni})][H])([G][Z(\theta_{10}, d_{10})]\ldots[Z(\theta_{n0}, d_{n0})][H])^{-1}.\end{aligned} \tag{2}$$

This can be viewed as

$$[D_{0i}] = [T(\Delta\theta_1, \mathsf{S}_1)]\ldots[T(\Delta\theta_n, \mathsf{S}_n)], \tag{3}$$

where

$$\begin{aligned}&[T(\Delta\theta_1, \mathsf{S}_1)] = [G][Z(\theta_{1i}, d_{1i})][Z(\theta_{10}, d_{10})]^{-1}[G]^{-1}, \\ &[T(\Delta\theta_2, \mathsf{S}_2)] = ([G][Z(\theta_{10}, d_{10})][X(\alpha_{12}, a_{12})][Z(\theta_{2i}, d_{2i})]) \\ &\qquad\qquad ([G][Z(\theta_{10}, d_{10})][X(\alpha_{12}, a_{12})][Z(\theta_{20}, d_{20})])^{-1}, \\ &\qquad\vdots \\ &[T(\Delta\theta_n, \mathsf{S}_n)] = ([G][[Z(\theta_{10}, d_{10})]\ldots) \\ &\qquad\qquad [Z(\theta_n, d_n)][Z(\theta_{n0}, d_{n0})]^{-1}([G][[Z(\theta_{10}, d_{10})]\ldots)^{-1}.\end{aligned} \tag{4}$$

The displacements $[T(\Delta\theta_i, \mathsf{S}_i)]$ are the relative rotations and translations along the joint axes of the robot from the chosen reference configuration. Notice that by expressing them in this way, the initial transformation $[G]$ is absorbed in the first joint axis and the final transformation $[H]$ disappears from the expression.

2.2 Dual Quaternions

The workspace of the robot can also be expressed by using the Clifford algebra of the *dual quaternions*. A spatial displacement can be represented as a dual quaternion,

$$\hat{Q}(\hat{\theta}) = \sin(\frac{\hat{\theta}}{2})\mathsf{S} + \cos(\frac{\hat{\theta}}{2}), \tag{5}$$

where $\mathsf{S} = \mathbf{s} + \varepsilon\mathbf{s}^0$, with $\varepsilon^2 = 0$, is the screw axis of the transformation. The dual numbers $\cos(\frac{\hat{\theta}}{2}) = \cos\frac{\theta}{2} + \varepsilon(-\frac{d}{2}\sin\frac{\theta}{2})$ and $\sin(\frac{\hat{\theta}}{2}) = \sin\frac{\theta}{2} + \varepsilon(\frac{d}{2}\cos\frac{\theta}{2})$ contain the information about the rotation about and the displacement along the screw axis. The components of the dual quaternions can be easily computed from the homogeneous matrix transformation.

The spatial displacements can be represented as the set of points $\mathbf{Z} = (\mathbf{Z}, \mathbf{Z}^0)$ in $\mathbf{R}^8$ which are subject to two constraints: $\mathbf{Z}\cdot\mathbf{Z} = 1$ and $\mathbf{Z}\cdot\mathbf{Z}^0 = 0$. Then the workspace can be represented as lying on a six-dimensional submanifold of $\mathbf{R}^8$.

The dual quaternion form for the kinematics equations of the robot are obtained by transforming eq.(4) into

$$\hat{D}^i = \hat{S}_1(\Delta\hat{\theta}^i_1)\ldots\hat{S}_n(\Delta\hat{\theta}^i_n), \tag{6}$$

where $\hat{D}^i$ is the dual quaternion for $[D_{0i}]$ and $\hat{S}_j$ is the dual quaternion for $[T(\Delta\theta_j, \mathsf{S}_j)]$.

This approach yields the kinematics equations as successive screw transformations from the reference position. It is a useful formulation from the synthesis point of view because the components of each axis appears explicitly in the base frame coordinates.

3 Synthesis of Constrained Serial Chains

Let $[T(\theta_1, \ldots, \theta_k)]$ be the kinematics equations of a serial robot, and let a discrete approximation of the desired workspace be given in the form of n goal transformations $[D_i], i = 0, \ldots, n-1$. The synthesis problem consists of solving the n matrix equations

$$[T(\theta_{1,i}, \ldots, \theta_{k,i})] = [D_i], \quad \imath = 0, \ldots, n-1. \tag{7}$$

We now transform these equations to successive screw displacements in dual quaternion form. The result is $n-1$ goal positions $\hat{D}^i, i = 1, \ldots, n-1$ and the kinematics equations $\hat{Q}(\hat{\theta}_1, \ldots, \hat{\theta}_k)$ to obtain the $n-1$ equations

$$\hat{Q}_i(\hat{\theta}^i_1, \ldots, \hat{\theta}^i_k) = \hat{D}^i, \quad i = 1, \ldots, n-1 \tag{8}$$

For each of the $n-1$ positions we define eight component equations. However, due to the structure of the dual quaternions, only six of them are independent. Remember for a unit dual quaternion, the 2-norm of the first vector is equal to one and the dot product of the first times the second is equal to zero.

Assume for the moment that the robot chain can be represented by an equivalent series of j revolute joints. Each of these joints has an axis which is defined by six Plucker coordinates, which yields $6j$ unknowns. The j joint variables take different values at each of the $n-1$ positions, which add $j(n-1)$ unknowns. This yields $6j + j(n-1)$ unknowns.

Two constraint equations are associated with Plucker coordinates arise for each joint axis. For each of the $n-1$ goal positions we obtain eight equations, which can be reduced to six. Thus, we have $2j + 6(n-1)$ equations.

Equating the number of unknowns to the number equations, we obtain

$$6j + j(n-1) = 6(n-1) + 2j. \tag{9}$$

Solving for n

$$n = \frac{3j+6}{6-j}, \tag{10}$$

we have that $2R$, $3R$, $4R$ and $5R$ spatial chains require 3, 5, 9, 21 positions, respectively. However, we need to consider some limitations. In eq. (9) we equate dual quaternions component by component. As the rotations operate independently in spatial displacements, the number of spherical positions we can reach will be limited by this fact, while the number of spatial translations is computed in general. Hence, to compute complete spatial positions, first we need to check how this are limited by the maximum number of spherical positions we can reach. To separate rotations from translations, assume our robot consists of l rotational joints and k translational joints. We therefore need two equations; the first one equating rotational joint directions with rotation components of the dual quaternion,

$$3l + l(n_R - 1) = 3(n_R - 1) + l \tag{11}$$

and the second equating both rotational and translational joints to the whole quaternion,

$$6(l+k) + (l+k)(n-1) = 6(n-1) + 2l + k \tag{12}$$

From the rotation equation,

$$n_R = \frac{3+l}{3-l} \tag{13}$$

Notice that this coincides with the results of the spherical robots: for one revolute joint we obtain finite number of solutions for two positions, this means we can reach one relative rotation. For two revolute joints we have finite number of solutions for $n_R = 5$, while for three we get infinity, which means that we can reach any orientation. The formula stops making sense after this. The maximum number of complete positions we can reach will be restricted by n_R, and if in the second formula we obtain more than that, the rest will be just translational components of dual quaternions in which rotations will have to be bounded to the given workspace.

Notice also that here we assume that the axes of the rotational and translational joints are not related, but it is easy to adapt the formula to particular cases in which the joints are constrained.

4 Solving the Design Equations

The design equations for constrained robots contain joints variables and axis variables. Our goal is to eliminate the joint variables, if possible, and solve for the axis variables. The axis variables define the physical dimensions of the robot.

In order to eliminate the joint parameters, we consider the equations for each position independently. This is called "implicitization" of the parametric equations. The first step in this implicitization process uses the semi-direct product structure of the group of spatial displacements, which isolates the product of rotations from translations. This behavior is captured in dual quaternion product, where the first four components never are mixed with the last four in any computation.

The four rotational components of the dual quaternion equation are parameterized only by the revolute joint variables,

$$\hat{Q}_{rot}(\theta_1,\dots,\theta_k)=\begin{Bmatrix} q_x(\theta_1,\dots,\theta_k)\\ q_y(\theta_1,\dots,\theta_k)\\ q_z(\theta_1,\dots,\theta_k)\\ q_w(\theta_1,\dots,\theta_k)\end{Bmatrix}=\begin{Bmatrix} p_x\\ p_y\\ p_z\\ p_w\end{Bmatrix} \tag{14}$$

This can only be transformed to a linear system that allows to solve for two of the revolute joint variables as a function of the joint axes and the rest of revolute variables,

$$\begin{Bmatrix} \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\\ \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\\ \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\\ \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\end{Bmatrix}=[R(\theta_3,\dots,\theta_k)]^T\begin{Bmatrix} p_x\\ p_y\\ p_z\\ p_w\end{Bmatrix} \tag{15}$$

where the matrix $[R(\theta_3,\dots,\theta_k)]$ is orthogonal for non-degenerated cases. Degenerated cases can be for instance solutions in which the serial chain is not spatial but planar.

This allows us to eliminate linearly two of the rotational parameters in the form of a vector of sine and cosines. We can then substitute these expressions in the second four components of the dual quaternion,

$$\begin{aligned}&\hat{Q}_{trans}(\theta_3,\dots,\theta_k,d_1,\dots,d_l)=\\ &=\begin{Bmatrix} q^0_x(\theta_3,\dots,\theta_k,d_1,\dots,d_l)\\ q^0_y(\theta_3,\dots,\theta_k,d_1,\dots,d_l)\\ q^0_z(\theta_3,\dots,\theta_k,d_1,\dots,d_l)\\ q^0_w(\theta_3,\dots,\theta_k,d_1,\dots,d_l)\end{Bmatrix}=\begin{Bmatrix} p^0_x\\ p^0_y\\ p^0_z\\ p^0_w\end{Bmatrix}\end{aligned} \tag{16}$$

To this equations we need to add any condition on the additional joint variables that is implicit in the solution for the rotations. The subsequent joint variables can be eliminated in a similar fashion. In the example below the process is illustrated for a TPR chain.

5 Synthesis of a Parallel TPR Robot

5.1 Synthesis equations for the TPR chain

The TPR serial chain is a four-degree of freedom robot. The base joint T consists of two revolute joints about perpendicular axes. This joint is also called U-joint for universal joint. The fixed axis G_1 allows rotation of angle θ_1 about it. Located at 90^o and intersecting G_1 is the second revolute axis, G_2, which allows rotation of angle θ_2. This is followed by a translation d along an arbitrary direction P and finally a rotation of angle ϕ about an arbitrary axis W, see Figure 1.

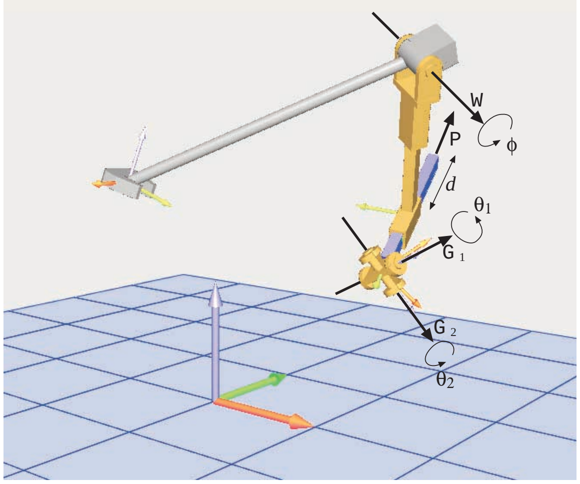

Figure 1. The spatial TPR robot

We call **c** to the intersection point of the two rotation axes G_1 and G_2. Notice that the location of the prismatic joint is immaterial and has been asigned in the drawing to the same intersection point.

The dual quaternion representation for the relative displacements of the chain is given by

$$\hat{Q}_{TPR}=\hat{G}_1(\theta_1,0)\hat{G}_2(\theta_2,0)\hat{P}(0,d)\hat{W}(\phi,0), \tag{17}$$

When applying the dual quaternion product we obtain the expression $\hat{Q}_{TPR} = Q^0 + \mathsf{Q}$, where the point is

$$\begin{aligned} Q^0 = {} & \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\cos\frac{\phi}{2} - \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\sin\frac{\phi}{2}\mathsf{G}_1\cdot\mathsf{W} \\ & - \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\frac{\phi}{2}\mathsf{G}_2\cdot\mathsf{W} - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\frac{\phi}{2}(\mathsf{G}_1\times\mathsf{G}_2)\cdot\mathsf{W} \\ & - \varepsilon\Big(\frac{d}{2}\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\cos\frac{\phi}{2}\mathsf{G}_1\cdot\mathsf{P} + \frac{d}{2}\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\phi}{2}\mathsf{G}_2\cdot\mathsf{P} \\ & + \frac{d}{2}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\sin\frac{\phi}{2}\mathsf{P}\cdot\mathsf{W} + \frac{d}{2}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\phi}{2}(\mathsf{G}_1\times\mathsf{G}_2)\cdot\mathsf{P} \\ & + \frac{d}{2}\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\sin\frac{\phi}{2}(\mathsf{G}_1\times\mathsf{P})\cdot\mathsf{W} + \frac{d}{2}\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\frac{\phi}{2}(\mathsf{G}_2\times\mathsf{P})\cdot\mathsf{W} \\ & + \frac{d}{2}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\frac{\phi}{2}((\mathsf{G}_1\times\mathsf{G}_2)\times\mathsf{P})\cdot\mathsf{W}\Big), \end{aligned} \tag{18}$$

and the dual vector

$$\begin{aligned} \mathsf{Q} = {} & \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\cos\frac{\phi}{2}\mathsf{G}_1 + \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\phi}{2}\mathsf{G}_2 \\ & + \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\sin\frac{\phi}{2}\mathsf{W} + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\phi}{2}\mathsf{G}_1\times\mathsf{G}_2 \\ & + \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\sin\frac{\phi}{2}\mathsf{G}_1\times\mathsf{W} + \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\frac{\phi}{2}\mathsf{G}_2\times\mathsf{W} \\ & + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\frac{\phi}{2}(\mathsf{G}_1\times\mathsf{G}_2)\times\mathsf{W} \\ & + \varepsilon\Big(\frac{d}{2}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\cos\frac{\phi}{2}\mathsf{P} \\ & + \frac{d}{2}\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\sin\frac{\phi}{2}((\mathsf{G}_1\times\mathsf{P})\times\mathsf{W} - (\mathsf{G}_1\cdot\mathsf{P})\mathsf{W}) \\ & + \frac{d}{2}\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\frac{\phi}{2}((\mathsf{G}_2\times\mathsf{P})\times\mathsf{W} - (\mathsf{G}_2\cdot\mathsf{P})\mathsf{W}) \\ & + \frac{d}{2}\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\cos\frac{\phi}{2}\mathsf{G}_1\times\mathsf{P} + \frac{d}{2}\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\phi}{2}\mathsf{G}_2\times\mathsf{P} \\ & + \frac{d}{2}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\cos\frac{\phi}{2}(\mathsf{G}_1\times\mathsf{G}_2)\times\mathsf{P} + \frac{d}{2}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\sin\frac{\phi}{2}\mathsf{P}\times\mathsf{W} \\ & + \frac{d}{2}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\sin\frac{\phi}{2}(((\mathsf{G}_1\times\mathsf{G}_2)\times\mathsf{P})\times\mathsf{W} - ((\mathsf{G}_1\times\mathsf{G}_2)\cdot\mathsf{P})\mathsf{W})\Big) \end{aligned} \tag{19}$$

The expansion of this equations componentwise leads to a set of equations in the components of the axes. The T-joint axis is formulated so that the coordinates of the intersection point **c** appear explicitely, as the point is also a design parameter,

$$\begin{aligned} \mathsf{G1} &= (g_{1x}, g_{1y}, g_{1z}) + \varepsilon((c_x, c_y, c_z)\times(g_{1x}, g_{1y}, g_{1z})) \\ \mathsf{G2} &= (g_{2x}, g_{2y}, g_{2z}) + \varepsilon((c_x, c_y, c_z)\times(g_{2x}, g_{2y}, g_{2z})) \end{aligned} \tag{20}$$

The moving prismatic axis has direction $\mathbf{P} = (p_x, p_y, p_z)$ and arbitrary location that will not appear in the design equations. The moving rotation axis is expressed in Plucker coordinates, $\mathsf{W} = (w_x, w_y, w_z) + \varepsilon(w_x^0, w_y^0, w_z^0)$.

The number of positions needed to obtain finite number of solutions is calculated as explained in the previous section. As we have three rotational joints, the robot will be able to reach any orientation and the orientation does not limit the number of complete positions to reach. We have $6+3+3+6+4(n-1)$ unknowns, corresponding to the direction $\mathbf{G}_1$, the point **c**, the direction $\mathbf{G}_2$, the direction **P** and the line W, plus the joint variables for the $n-1$ relative transformations. We have $1+2+1+2+6(n-1)$ equations, corresponding to the unit vector conditions for all directions, the perpendicularity condition between $\mathbf{G}_1$ and $\mathbf{G}_2$ and the moment condition for W, plus the six independent equations of equating to the goal dual quaternion . Therefore we need to define $n = 7$ positions.

To create the design equations we equate the expanded eqs.(18, 19) to the goal dual quaternion $\hat{D}$, that is,

$$\hat{Q}_{TPR}(\theta_1, \theta_2, d, \phi) - \hat{D}^i = \vec{0}, \tag{21}$$

to obtain one of the sets of design equations. After equating for all the relative dual quaternion transformations, we obtain six sets of dual quaternion equations. However, to eliminate the joint parameters we work with only a generic set and apply the results to each relative position.

To eliminate the joint parameters we consider the separation between rotations and translations. It is easy to solve for two of the rotational joint parameters. Every direction will be reached by moving the rotation axes accordingly to the third rotation parameter as appears in the solution of the linear system,

$$[R(\phi)]\begin{Bmatrix} \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2} \\ \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} \\ \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} \\ \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} \end{Bmatrix} = \begin{Bmatrix} p_x \\ p_y \\ p_z \\ p_w \end{Bmatrix} \tag{22}$$

with

$$[R(\phi)] = \begin{bmatrix} \cos\frac{\phi}{2}\mathbf{g}_2 + \sin\frac{\phi}{2}\mathbf{g}_2\times\mathbf{w} & \cos\frac{\phi}{2}\mathbf{g}_1 + \sin\frac{\phi}{2}\mathbf{g}_1\times\mathbf{w} & \cos\frac{\phi}{2}\mathbf{g}_1\times\mathbf{g}_2 + \sin\frac{\phi}{2}((\mathbf{g}_1\times\mathbf{g}_2)\times\mathbf{w} - (\mathbf{g}_1\cdot\mathbf{g}_2)\mathbf{w}) & \sin\frac{\phi}{2}\mathbf{w} \\ -\sin\frac{\phi}{2}\mathbf{g}_2\cdot\mathbf{w} & -\sin\frac{\phi}{2}\mathbf{g}_1\cdot\mathbf{w} & -\cos\frac{\phi}{2}\mathbf{g}_1\cdot\mathbf{g}_2 - \sin\frac{\phi}{2}(\mathbf{g}_1\times\mathbf{g}_2)\cdot\mathbf{w} & \cos\frac{\phi}{2} \end{bmatrix} \tag{23}$$

In equation (24) we show the two first columns of this matrix.

$$[R(\phi)] = \begin{bmatrix} g_{2x}\cos\frac{\phi}{2} + (g_{2y}w_z - g_{2z}w_y)\sin\frac{\phi}{2} & g_{1x}\cos\frac{\phi}{2} + (g_{1y}w_z - g_{1z}w_y)\sin\frac{\phi}{2} & \dots & \dots \\ g_{2y}\cos\frac{\phi}{2} + (g_{2z}w_x - g_{2x}w_z)\sin\frac{\phi}{2} & g_{1y}\cos\frac{\phi}{2} + (g_{1z}w_x - g_{1x}w_z)\sin\frac{\phi}{2} & \dots & \dots \\ g_{2z}\cos\frac{\phi}{2} + (g_{2x}w_y - g_{2y}w_x)\sin\frac{\phi}{2} & g_{1z}\cos\frac{\phi}{2} + (g_{1x}w_y - g_{1y}w_x)\sin\frac{\phi}{2} & \dots & \dots \\ -(g_{2x}w_x + g_{2y}w_y + g_{2z}w_z)\sin\frac{\phi}{2} & -(g_{1x}w_x + g_{1y}w_y + g_{1z}w_z)\sin\frac{\phi}{2} & \dots & \dots \end{bmatrix} \tag{24}$$

The matrix $[R(\phi)]$ is an orthogonal matrix. The solution for the T-joint angles is

$$\begin{Bmatrix} \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2} \\ \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} \\ \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} \\ \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} \end{Bmatrix} = [R(\phi)]^T \begin{Bmatrix} p_x \\ p_y \\ p_z \\ p_w \end{Bmatrix} \tag{25}$$

The solution always exists for directions $\mathbf{g}_1$, $\mathbf{g}_2$ $\mathbf{w}$ and angles ϕ such that the system is solvable, which we can assume will be given by the solution of the design equations. In this case there is not planar degeneracy if we consider the constraint for $\mathbf{g}_1$ and $\mathbf{g}_2$ to be at right angles. The angle ϕ is constrained by the relation among the four variables we are solving for,

$$(\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}+\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2})^2+(\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}-\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2})^2=1, \tag{26}$$

obtaining the condition for ϕ,

$$A_0\cos^2\frac{\phi}{2}+B_0\sin^2\frac{\phi}{2}+C_0\cos\frac{\phi}{2}\sin\frac{\phi}{2}+D_0=0. \tag{27}$$

The solutions for the θ_1, θ_2 angles are substituted in the four moment equations of the dual quaternion. We obtain four equations which are linear in the joint translation d and quadratic in the joint rotation ϕ, and that we denote by

$$\begin{aligned} &(A_{1i}d+A_{0i})\cos^2\frac{\phi}{2}+(B_{1i}d+B_{0i})\sin^2\frac{\phi}{2}+ \\ &(C_{1i}d+C_{0i})\cos\frac{\phi}{2}\sin\frac{\phi}{2}+D_{0i}=0, \quad i=1,\dots,4 \end{aligned} \tag{28}$$

To eliminate ϕ, we add the previously obtained angle condition, eq. (27), to create the homogeneous system

$$\begin{bmatrix} A_{11}d+A_{01} & B_{11}d+B_{01} & C_{11}d+C_{01} & D_{01} \\ \vdots & \vdots & \vdots & \vdots \\ A_{14}d+A_{04} & B_{14}d+B_{04} & C_{14}d+C_{04} & D_{04} \\ A_0 & B_0 & C_0 & D_0 \end{bmatrix} \begin{Bmatrix} \cos^2\frac{\phi}{2} \\ \sin^2\frac{\phi}{2} \\ \cos\frac{\phi}{2}\sin\frac{\phi}{2} \\ 1 \end{Bmatrix} = \vec{0} \tag{29}$$

For the system to have solutions, all five 4×4 minors must be equal to zero. Obviously the first minor, which we call M_5 and corresponds to the determinant of the first four rows (eliminating row 5), is equal to zero by construction of the dual quaternions.

We can obtain the subspace of solutions from the matrix corresponding to the first four rows, as the determinant M_5 is always zero. By solving linearly in this system for the variables $\cos^2\frac{\phi}{2}$, $\sin^2\frac{\phi}{2}$ and $\cos\frac{\phi}{2}\sin\frac{\phi}{2}$, we obtain expressions as a function of the prismatic joint variable d. The relation between these three solutions lead to two equations in d, namely

$$\begin{aligned} &\cos^2\frac{\phi}{2}+\sin^2\frac{\phi}{2}=1 \\ &(\cos^2\frac{\phi}{2})(\sin^2\frac{\phi}{2})=(\cos\frac{\phi}{2}\sin\frac{\phi}{2})^2. \end{aligned} \tag{30}$$

The first of these equalities leads to a cubic equation in d and the second is a quartic equation in d. To these two conditions, we must add another cubic in d,

$$K_{3i}d^3+K_{2i}d^2+K_{1i}d+K_{0i}=0, \quad i=1,\dots,4 \tag{31}$$

which appears as any of the minors in the matrix of eq.(**??**), which we call M_i for $i=1,\dots,4$. All four minors obtained in this fashion are in fact the same cubic equation in d.

Out of the system of three equations in d, we can eliminate the parameter d,

$$\begin{bmatrix} 0 & K_{31} & K_{21} & K_{11} & K_{01} \\ 0 & K_{32} & K_{22} & K_{12} & K_{02} \\ K_{43} & K_{33} & K_{23} & K_{13} & K_{03} \end{bmatrix} \begin{Bmatrix} d^4 \\ d^3 \\ d^2 \\ d \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{32}$$

to obtain two design equations per goal dual quaternion, which are free of joint variables and depend only on the coordinates of the joint axes. These 12 equations together with 6 constraints,

$$\begin{aligned} &g_{1x}^2+g_{1y}^2+g_{1z}^2=1, \\ &g_{2x}^2+g_{2y}^2+g_{2z}^2=1, \quad g_{1x}g_{2x}+g_{1y}g_{2y}+g_{1z}g_{2z}=0 \\ &w_x^2+w_y^2+w_z^2=1, \quad w_xw_x^0+w_yw_y^0+w_zw_z^0=0 \\ &p_x^2+p_y^2+p_z^2=1 \end{aligned} \tag{33}$$

allows us to solve for the 18 unknowns corresponding to the four joint axes.

5.2 Synthesis of the parallel 2-TPR robot

From the design equations for the TPR serial chain, we will obtain a certain number of solutions. If we naively join the end-link of two of them, we will create a two degree of freedom parallel robot, the 2-TPR robot. The 2-TPR robot will reach the set of seven positions, but nothing ensures that the movement of the robot will be smooth or even possible. There are two possible approaches to deal with this situation. One is to incorporate a series of conditions in the synthesis process. These conditions include interference, singularities, branches, joint limits and some others. In this approach, we transform the solution procedure to a constrained optimization problem. The second is to analyze the set of solutions after the synthesis and pick the ones that agree with the imposed conditions.

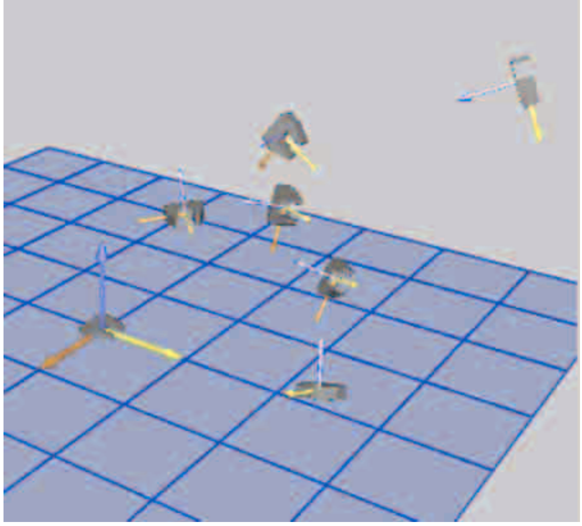

Figure 2. The seven initial positions

6 Example

We present an example in which we want to design the TPR robot to reach seven positions. To pick seven arbitrary positions in space is a difficult task; we can use two strategies to help in the choice. One option is to define an initial position, an intermediate position and a final position and perform dual quaternion interpolation with two intermediate positions (19). The TPR robot will exactly hit the seven positions in the trajectory. The second option, is to set some of the parameters of the TPR chain to desired values and solve for a smaller number of positions.

In this case we solve for the seven position for the first chain, and for the second chain we set both the directions of the rotations of the T-joint $\mathbf{g}_1$ and $\mathbf{g}_2$, which is equivalent to fix the plane of the rotation, to a specified value, and we also impose the condition that the moving revolute joint axis W must be perpendicular to the prismatic joint direction $\mathbf{P}$. Using the counting formula, we see that we can solve for a finite number of solutions for $n = 5$ positions.

Table 1. THE GOAL POSITIONS

Pos.	*Axis*	*Rot.*	*Trans.*
pos. 1	$(1.0, 0.0, 0.0) + \varepsilon(0.0, 0.0, 0.0)$	0^{o}	0
pos. 2	$(0.5, 0.8, -0.4) + \varepsilon(-1.8, 0.8, -0.7)$	68.9^{o}	0.32
pos. 3	$(-0.2, 0.9, -0.3) + \varepsilon(-1.7, -0.3, 0.2)$	92.7^{o}	0.71
pos. 4	$(0.0, 0.8, -0.5) + \varepsilon(-2.2, 0.0, 0.2)$	156.5^{o}	1.39
pos. 5	$(0.3, 0.9, -0.3) + \varepsilon(-1.6, 0.5, -0.1)$	79.0^{o}	0.31

Table 2. THE JOINT AXES FOR FIRST TPR CHAIN

Joint Axis	*Direction*	*Moment*
G_1	$(0.52, 0.34, -0.78)$	$(-1.39, 1.06, -0.47)$
G_2	$(-0.41, 0.90, 0.11)$	$(-0.72, -0.48, 1.20)$
P	$(0.81, 0.46, 0.35)$	$(0.02, 0.54, -0.77)$
W	$(0.48, 0.86, -0.19)$	$(-1.83, 0.69, -1.49)$

Table 3. THE JOINT AXES FOR SECOND TPR CHAIN

Joint Axis	*Direction*	*Moment*
G_1	$(1.0, 0.0, 0.0)$	$(0.0, 0.98, -2.15)$
G_2	$(0.0, 1.0, 0.0)$	$(-0.98, 0.0, 1.0)$
P	$(-0.68, -0.33, 0.66)$	$(1.74, -1.33, 1.12)$
W	$(0.49, -0.87, 0.08)$	$(1.29, 0.58, -1.75)$

On Table 1 and Figure 2 show the specified positions,

The design equations are very sensitive to the initial conditions, and in this particular case we could not find any solution for the second chain and the numerical solver led to a chain that hits four of the five positions. In Table(2, 3) we can see the obtained solutions. Figure 3 shows the parallel 2-TPR robot while reaching positions 1, 2 and 5.

7 Conclusions

This paper introduces a new formulation for the exact finite position kinematic synthesis of constrained serial robots which can be applied to the design of spatial parallel robots due to its broader applicability. The arbitrary serial chains can have up to six degrees of freedom. The standard kinematics equations of the chain is transformed into successive screw displacements and then expressed in dual quaternions. The result is an explicit set of axis parameters that define the robot and an additional set of joint parameters that can be eliminated. The structure of these equations provide a convenient strategy for this elimination, which present for the spatial TPR robot. The parallel 2-TPR robot is constructed by joining the end-links of two obtained solutions. Future work include the improvement of the numerical behavior of the design equations and the addition of constraints to ensure the smooth movement of the parallel robot.

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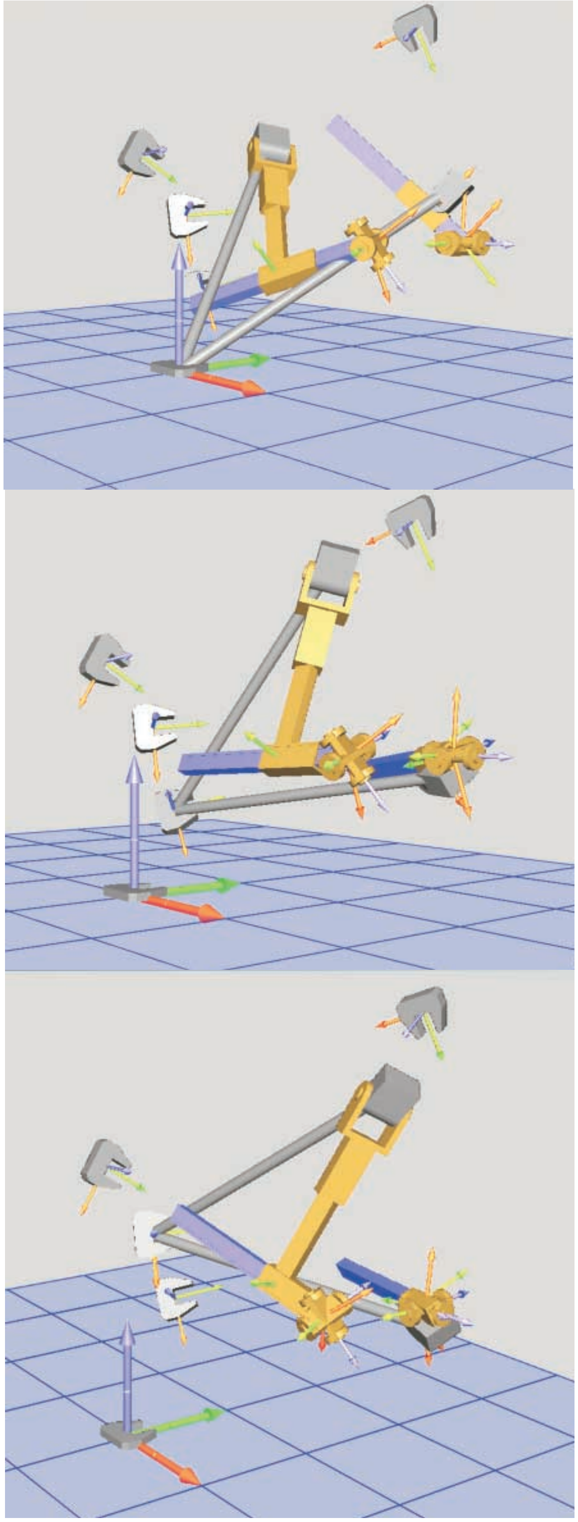

Figure 3. The 2-TPR robot reaching positions 1, 2 and 5