Dimensional Synthesis of Robots using a Double Quaternion Formulation of the Workspace

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Abstract

This paper presents a synthesis procedure for a robot that guides an end-effector as close as possible to a user-specified trajectory. The technique maps the goal workspace from the group of spatial displacements, $SE(3)$, to the group of 4×4 rotations, $SO(4)$, in order to obtain a bi-invariant metric for the design procedure. Double quaternions are used to provide a convenient parameterization for $SO(4)$. An example is presented that compares designs obtained using dual quaternions to those for double quaternions, for varying locations of the fixed frame, in order to demonstrate the technique.

1. Introduction

The design of a robot can be viewed as fitting its workspace to a specifed task manifold. The use of optimization procedures to minimize the error between these two manifolds must address the fact that distance measurments between positions of a rigid body in space are not coordinate frame invariant [1].

Our approach transforms both the workspace and task manifolds from the group of rigid displacements in three dimensional space, $SE(3)$, to the group of rigid rotations in four dimensional space, $SO(4)$. This later group has a coordinate invariant metric, and a convenient mathematical structure, that provides a useful framework for our design optimization algorithm.

Ge [2] and Etzel and McCarthy [3] developed the basic approach to the mapping $SE(3)$ to $SO(4)$ and introduced "biquaternions" to facilitate the calculations. These hypercomplex numbers are now called "double quaternions," and in this paper we develop the double quaternion form of the kinematics equations of a robot. This formulation becomes the foundation of our design optimization procedure.

2. Double Quaternions

Here we introduce the relationship between spatial displacements and rotations in four dimensions. The construction that yields dual quaternions from 4×4 homogeneous transforms [4], yields double quaternions when applied to these rotation matrices.

2.1. Homogeneous transforms

A spatial displacement consists of a 3×3 rotation matrix, $[A]$, and a 3×1 translation vector, $\mathbf{d}$, assembled into a 4×4 homogeneous transform. For the specific case of a screw displacement of angle θ around the z-axis and slide d along it, we have the 4×4 transform

$$[Z(\theta, d)] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (1)$$

This matrix defines a rigid displacement in the three-dimensional hyperplane $x_4 = 1$ of four dimensional Euclidean space. The same displacement can be defined in parallel hyperplanes, $x_4 = R$, by simply dividing the translation terms by R, so we have

$$[Z(\theta, d)] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & \frac{d}{R} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (2)$$

This form of the homogeneous transform is easily extended to a 4×4 rotation.

2.2. 4×4 rotations

Rather than develop the equation for a general 4×4 rotation, we focus on the rotation $[Z(\theta, \gamma)]$ that extends the screw displacement $[Z(\theta, d)]$. Similar rotations can be obtained that extend screw displacements along the x and y axes.

Consider the 4×4 rotation matrix,

$$[Z(\theta,\gamma)] = \begin{bmatrix}\cos\theta & -\sin\theta & 0 & 0\\ \sin\theta & \cos\theta & 0 & 0\\ 0 & 0 & \cos\gamma & \sin\gamma\\ 0 & 0 & -\sin\gamma & \cos\gamma\end{bmatrix}. \quad (3)$$

Notice that if γ is defined such that

$$\tan\gamma = \frac{d}{R}, \quad (4)$$

then for R large enough (3) approximates (2). The difference is in the $(4,3)$ element which is orthogonal to the $x_4 = R$ hyperplane of the rigid displacement. We consider the 4×4 rotation $[Z(\theta,\gamma)]$ to be an extension of the screw displacement $[Z(\theta,d)]$.

2.3. Double matrices

A 4×4 rotation matrix, $[R]$, operates on a point, $\mathbf{x} = (x_1, x_2, x_3, x_4)^T$, in a moving frame M to compute its coordinates, $\mathbf{X} = (X_1, X_2, X_3, X_4)^T$, in the fixed frame F. A pair of points $\mathbf{x}$ and $\mathbf{y}$ defines a two-dimensional subspace that has the coordinate bivector, $\mathbf{x}\wedge\mathbf{y} = (p_{41}, p_{42}, p_{43}, p_{23}, p_{31}, p_{12})^T$, where

$$p_{ij} = x_i y_j - x_j y_i, \quad i \neq j = 1,2,3,4. \quad (5)$$

From the coordinate transformations $\mathbf{X} = [R]\mathbf{x}$ and $\mathbf{Y} = [R]\mathbf{y}$, it is possible to directly compute the 6×6 matrix, $[\tilde{R}]$, that transforms a bivector $\mathbf{x}\wedge\mathbf{y}$ in M to the bivector $\mathbf{X}\wedge\mathbf{Y}$ in F.

For the rotation matrix, $[Z(\theta,\gamma)]$, we obtain

$$[\tilde{Z}(\theta,\gamma)] = \begin{bmatrix}A & B\\ B & A\end{bmatrix}, \quad (6)$$

where

$$A = \begin{bmatrix}\cos\gamma\cos\theta & -\cos\gamma\sin\theta & 0\\ \cos\gamma\sin\theta & \cos\gamma\cos\theta & 0\\ 0 & 0 & 1\end{bmatrix} \quad (7)$$

and

$$B = \begin{bmatrix}-\sin\gamma\sin\theta & -\sin\gamma\cos\theta & 0\\ \sin\gamma\cos\theta & -\sin\gamma\sin\theta & 0\\ 0 & 0 & 0\end{bmatrix} \quad (8)$$

The structure of this 6×6 matrix allows us to introduce the symbol, j, with the property that $j^2 = 1$, in order to construct the "double vector" $\tilde{\mathbf{p}} = (p_{41} + jp_{23}, p_{42} + jp_{31}, p_{43} + jp_{12})^T$, and the "double matrix" $[\tilde{Z}] = [A + jB]$, so

$$\tilde{\mathbf{P}} = [\tilde{Z}]\tilde{\mathbf{p}}. \quad (9)$$

The double matrix $[\tilde{Z}] = [A + jB]$ can be simplified to a more suggestive form by noting that the symbol j satisfies the identities:

$$\cos(j\gamma) = \cos(\gamma), \quad \text{and} \quad \sin(j\gamma) = j\sin(\gamma), \quad (10)$$

obtained from the series expansions of the sine and cosine functions. The result is that $[\tilde{Z}] = [A + jB]$ becomes:

$$[\tilde{Z}] = \begin{bmatrix}\cos(\theta + j\gamma) & -\sin(\theta + j\gamma) & 0\\ \sin(\theta + j\gamma & \cos(\theta + j\gamma) & 0\\ 0 & 0 & 1\end{bmatrix}. \quad (11)$$

The "double quaternion" associated with this matrix is $\tilde{Z}(\theta,\gamma)$, given by:

$$\tilde{Z}(\theta,\gamma) = (0, 0, \sin\frac{\theta + j\gamma}{2}, \cos\frac{\theta + j\gamma}{2})^T. \quad (12)$$

An identical calculation can be performed using the matrix $[X(\alpha,\rho)]$ consisting of a rotation by α in the $y - z$ plane, and rotation by ρ in the $x - w$ plane. The result is the double matrix,

$$[\tilde{X}] = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos(\alpha + j\rho) & -\sin(\alpha + j\rho)\\ 0 & \sin(\alpha + j\rho) & \cos(\alpha + j\rho)\end{bmatrix}, \quad (13)$$

and the double quaternion,

$$\tilde{X}(\alpha,\rho) = (\sin\frac{\alpha + j\rho}{2}, 0, 0, \cos\frac{\alpha + j\rho}{2})^T. \quad (14)$$

3. Double Kinematics Equations

The Denavit-Hartenberg convention allows us to write the kinematics equations of a robot in the standard form:

$$[D] = [G][Z(\theta_1, d_1)][X(\alpha_{12}, a_{12})]\ldots [X(\alpha_{56}, a_{56})][Z(\theta_6, d_6)][H]. \quad (15)$$

The matrices $[G]$ and $[H]$ locate the base of the robot in F and the moving frame M in the end-effector.

Extending each of these 4×4 homogeneous transforms to 4×4 rotation matrices and converting to double quaternions, we obtain the "double kinematics equations:"

$$\tilde{D} = \tilde{G}\tilde{Z}(\theta_1, \gamma_1)\tilde{X}(\alpha_{12}, \rho_{12})\ldots \tilde{X}(\alpha_{56}, \rho_{56})\tilde{Z}(\theta_6, \gamma_6)\tilde{H}. \quad (16)$$

The double quaternion $\tilde{D} = D_1 + jD_2$ can be reformulated in useful way, by introducing the parameters $\xi = (1 + j)/2$ and $\eta = (1 - j)/2$ to yield,

$$\tilde{D} = (D_1 + D_2)\xi + (D_1 - D_2)\eta. \quad (17)$$

Notice that because $\xi^2 = \xi$, $\eta^2 = \eta$ and $\xi\eta = 0$ the terms $(D_1 + D_2)$ and $(D_1 - D_2)$ operate independently in the double quaternion product. In the case of the double quaternions $\tilde{Z}(\theta, \gamma)$ and $\tilde{X}(\alpha, \rho)$ we have:

$$\tilde{Z}(\theta,\gamma) = \begin{Bmatrix} 0 \\ 0 \\ \sin\frac{\theta+\gamma}{2} \\ \cos\frac{\theta+\gamma}{2} \end{Bmatrix} \xi + \begin{Bmatrix} 0 \\ 0 \\ \sin\frac{\theta-\gamma}{2} \\ \cos\frac{\theta-\gamma}{2} \end{Bmatrix} \eta, \qquad (18)$$

and

$$\tilde{X}(\alpha,\rho) = \begin{Bmatrix} \sin\frac{\alpha+\rho}{2} \\ 0 \\ 0 \\ \cos\frac{\alpha+\rho}{2} \end{Bmatrix} \xi + \begin{Bmatrix} \sin\frac{\alpha-\rho}{2} \\ 0 \\ 0 \\ \cos\frac{\alpha-\rho}{2} \end{Bmatrix} \eta, \qquad (19)$$

This allows us to separate the double kinematics equations into two independent sets of component quaternion equations, which are manipulated separately. This is our preferred form of the double quaterion.

4. The Design Process

Dimensional synthesis theory provides a way to compute the dimensions of an open chain that reaches a finite set of positions [5]. In order to demonstrate our design approach, we determine the set of all open chains that reach finite subsets of positions along a specified trajectory, and select the one that minimizes the error in reaching all of the positions. This selection process is necessarily coordinate frame dependent. The goal is to demonstrate the impact of the double quaternion formulation on reducing this coordinate frame dependence.

4.1. The specified trajectory

For our examples, we interpolate six arbitrarily specified positions and obtain 16 individual instances of the position of the end-effector along its trajectory. We use the Bezier interpolation strategy of Ge and Ravani [6] to generate the sequence of positions under the control of a set of key frames; also see [7]. Table 1 lists the key frames used for these examples, and Figs. 1 and 2 show two versions of the resulting trajectory.

4.2. CC chain synthesis

Consider the four degree of freedom CC chain shown in Fig. 3. It is a simple matter to determine the fixed axis, $\mathsf{G} = (\mathbf{G}, \mathbf{B} \times \mathbf{G})$, associated with a moving fixed axis, $\mathsf{W} = (\mathbf{W}, \mathbf{P} \times \mathbf{W})$, that ensures that the chain can reach three arbitrarily specified positions.

Table 1: The key frame data used to specify an end-effector trajectory.

No.	d_x	d_y	d_z	Y-rot.	X-rot.	Z-rot
1	0	0	0	-90	45	0
2	1	1	2	-45	30	0
3	3	2	4	0	15	0
4	5	3	3	30	0	15
5	6.5	2	2	45	20	15
6	8	0	1	90	40	0

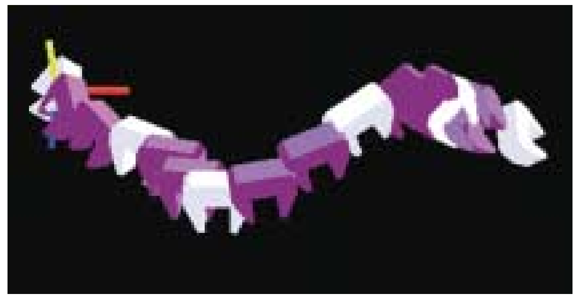

Figure 1: The end-effector trajectory that interpolates the dual quaternion coordinates of six key-frames.

Let the positions be defined by the three 4×4 homogeneous transforms, $[T_i]$, $[T_j]$ and $[T_k]$. We construct the two relative transformations $[T_{ij}] = [T_j][T_i^{-1}]$ and $[T_{ik}] = [T_k][T_i^{-1}]$ and solve the constraint equations of the chain.

4.2.1. *The direction equations*

The direction, $\mathbf{G}$, of the fixed axis is obtained from the relative rotations $[A_{ij}]$ and $[A_{ik}]$ by solving the pair of equations:

$$\mathbf{G}^T \begin{bmatrix} [A_{ij} - I] \\ [A_{ik} - I] \end{bmatrix} \mathbf{W} = 0, \qquad (20)$$

Figure 2: The same end-effector trajectory but interpolated using double quaternion coordinates of six key-frames.

which is seen to be

$$\mathbf{G} = k([A_{ij} - I]\mathbf{W}) \times ([A_{ik} - I]\mathbf{W}). \quad (21)$$

The coefficient k normalizes the magnitude of this vector to one.

4.2.2. The moment equations

Let moment terms of the fixed and moving axes be denoted $\mathbf{R} = \mathbf{B}\times\mathbf{G}$ and $\mathbf{V} = \mathbf{P}\times\mathbf{W}^1$, then, the moment constraint equations for the CC chain yield the pair of equations

$$\begin{Bmatrix}\mathbf{G}\\ \mathbf{R}\end{Bmatrix}^T \begin{bmatrix}\begin{bmatrix}D_{ij}A_{ij} & A_{ij}-I\\ A_{ij}-I & 0\end{bmatrix}\\ \begin{bmatrix}D_{ik}A_{ik} & A_{ik}-I\\ A_{ik}-I & 0\end{bmatrix}\end{bmatrix} \begin{Bmatrix}\mathbf{W}\\ \mathbf{V}\end{Bmatrix} = 0. \quad (22)$$

Note D_{ij} is the skew-symmetric matrix constructed from elements of the relative translation vector $\mathbf{d}_{ij}$ such that $D_{ij}\mathbf{y} = \mathbf{d}_{ij} \times \mathbf{y}$. Because $\mathbf{G}$ is known from the direction equations, this can be solved explicitly to determine moment $\mathbf{R} = \mathbf{B} \times \mathbf{G}$. The result is a unique CC chain for every set of three positions in the specified trajectory.

It is useful to note here that synthesis equations such as these exist for most open chains that often can be solved analytically for a finite number of task positions [5, 8].

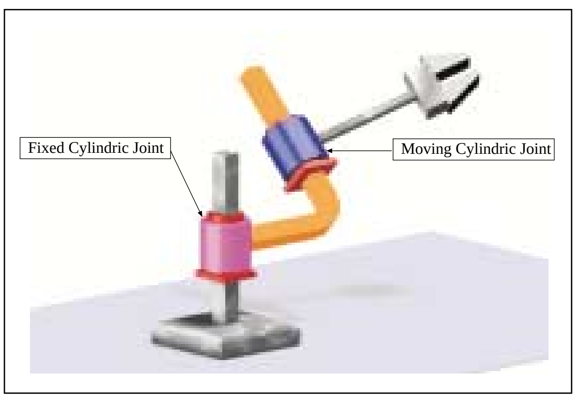

Figure 3: A spatial CC robot.

4.3. The optimization problem

We now introduce a measure of distances between positions defined in $SE(3)$ and rotations in $SO(4)$. The optimum design will be selected to minimize the sum of the square of these distances summed over all the positions in the specified trajectory.

4.3.1. Metrics for $SE(3)$ and $SO(4)$

Let $\hat{P} = P_1 + \epsilon P_2$ and $\hat{Q} = Q_1 + \epsilon Q_2$ be the dual quaternions associated with two 4×4 transforms in $SE(3)$ [4], then the distance between these positions is defined to be the magnitude of the eight-vector $\Delta = (\mathbf{P}_1, \mathbf{P}_2) - (\mathbf{Q}_1, \mathbf{Q}_2)$:

$$\text{dist}_E(\hat{P}, \hat{Q}) = \sqrt{(\mathbf{P}_1 - \mathbf{Q}_1)^2 + (\mathbf{P}_2 - \mathbf{Q}_2)^2}. \quad (23)$$

This metric is necessarily dependent on the coordinate frames, F and M, chosen to define the displacement.

Similarly, we define the distance between two 4×4 rotations to be the magnitude of the eight-vector $\Delta = (\mathbf{P}_1, \mathbf{P}_2) - (\mathbf{Q}_1, \mathbf{Q}_2)$ formed from their respective double quaternions $\tilde{P} = P_1\xi + P_2\eta$ and $\tilde{Q} = Q_1\xi + Q_2\eta$, that is

$$\text{dist}_O(\tilde{P}, \tilde{Q}) = \sqrt{(\mathbf{P}_1 - \mathbf{Q}_1)^2 + (\mathbf{P}_2 - \mathbf{Q}_2)^2}. \quad (24)$$

It is important to note that our preferred form for a double quaternion uses the units ξ and η, however, if instead the form $\tilde{P} = P_1' + jP_2'$ is used, then it is easy to obtain

$$\text{dist}_O(\tilde{P}, \tilde{Q}) = \sqrt{2}\sqrt{(\mathbf{P}_1' - \mathbf{Q}_1')^2 + (\mathbf{P}_2' - \mathbf{Q}_2')^2}. \quad (25)$$

This metric is invariant to changes of both in the fixed and moving coordinate frames [3].

4.3.2. The overall error

For our example, we have 16 frames along the desired trajectory of the end-effector, and we can determine a unique CC chain for each set of three frames. The result is a set of $\binom{16}{3} = 560$ designs. Our optimum design minimizes the sum of the square distances between the end-effector and each of the 16 target frames, which for each of the two ways we measure distances, is given by

$$S_E = \sum_{i=1}^{16} \text{dist}_E^2(\hat{P}_i, \hat{D}_i), \text{ and } S_O = \sum_{i=1}^{16} \text{dist}_O^2(\tilde{P}_i, \tilde{D}_i). \quad (26)$$

The $\hat{P}_i$ and $\tilde{P}_i$ are the dual and double quaternions associated with the specified positions along the trajectory. The $\hat{D}_i$ and $\tilde{D}_i$ are the dual and double quaternions representing the closest approach of the end-effector of the CC chain to each of these goal positions.

Table 2: The fixed axis is the line $L = \mathbf{B} + t\mathbf{G}$. Ex. 1, 2, 3 use dual quaternions, and Ex. 4, 5, 6 use double quaternions.

Ex.	F	$\mathbf{G}$	$\mathbf{B}$	Error
1	0	(-0.37, 0.91, 0.18)	(5.4, 1.7, 2.8)	0.288
2	+8	(0.19, 0.95, 0.26)	(0.72, -11.1, -3.0)	0.879
3	-8	(-0.35, 0.93, 0.04)	(3.0, 6.5, 1.1)	0.473
4	0	(0.05, 0.99, 0.11)	(3.8, -0.15, -0.29)	0.047
5	+8	(0.03, 0.99, 0.11)	(3.9, -0.15, -0.54)	0.047
6	-8	(0.05, 0.99, 0.11)	(3.8, -0.15, -0.28)	0.047

4.3.3. *The $\hat{D}_i$ and $\tilde{D}_i$*

We determine the closest approach dual quaternions $\hat{D}_i$ in $SE(3)$ using the dual quaternion form of the kinematics equations for the CC chain:

$$\hat{D} = \hat{G}\hat{Z}(\theta_1, d_1)\hat{X}(\alpha_{12}, a_{12})\hat{Z}(\theta_2, d_2)\hat{H}. \quad (27)$$

Since $\hat{G}$, $\hat{H}$ and α_{12} and a_{12} are known for a given design, the problem reduces to determining the joint parameter values, θ_1, θ_2, d_1 and d_2 that minimize $\text{dist}^2_E(\hat{P}, \hat{D})$. This is done by setting the derivative of this function to zero. Let $\mathbf{P}$ and $\mathbf{D}$ be the eight-vectors formed from the components of the respective dual quaternions, so we obtain

$$\left[\frac{\partial \mathbf{D}}{\partial \theta_1} \quad \frac{\partial \mathbf{D}}{\partial d_1} \quad \frac{\partial \mathbf{D}}{\partial \theta_2} \quad \frac{\partial \mathbf{D}}{\partial d_2}\right]^T (\mathbf{P} - \mathbf{D}) = 0. \quad (28)$$

We obtained algebraic solutions to these equations using Maple to determine the values of θ_1, θ_2, d_1 and d_2. Substitution of these values into the kinematics equations for the chain yields the dual quaternion $\hat{D}_i$ that comes the closest to $\hat{P}_i$.

A similar procedure yields the double quaternion $\tilde{D}_i$ in $SO(4)$ that comes the closest to the known frame $\tilde{P}_i$. In this case, we use the double quaternion form of the kinematics equations for the CC chain:

$$\tilde{D} = \tilde{G}\tilde{Z}(\theta_1, \gamma_1)\tilde{X}(\alpha_{12}, \rho_{12})\tilde{Z}(\theta_2, \gamma_2)\tilde{H}. \quad (29)$$

As above $\tilde{G}$, $\tilde{H}$ and α_{12} and ρ_{12} are known, and we find the values of the joint parameters, $\theta_1, \gamma_1, \theta_2, \gamma_2$, by minimizing the function $\text{dist}^2_O(\tilde{P}_i, \tilde{D}_i)$. This requires the solution of double quaternion equations identical in form to those given in (28). Algebraic solutions were obtained using Maple and substuted into the double kinematics equations to obtain $\tilde{D}_i$.

5. Example CC chain design

The six key frames used to define the trajectory of the end-effector are listed in Table 1. Figure 1 shows the six key frames and the additional 10 interpolated frames computed using Bezier interpolation for dual quaternion coordinates [6, 7]. Figure 2 presents the same interpolation computed using double quaternion coordinates. The axis $L_w = \vec{k} + 2t\vec{\jmath}$ located in the end-effector frame was selected for the moving axis, w, of the CC chain; thus $\mathsf{W} = [T_1]\mathsf{w}$ is used in the synthesis equations above to determine the fixed axis $\mathsf{G} = (\mathbf{G}, \mathbf{B} \times \mathbf{G})$ for each set of three positions in the specified trajectory.

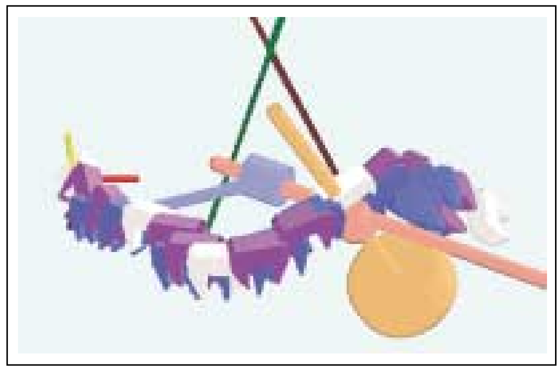

Figure 4: The end-effector trajectory traced by a CC-chain optimized using dual quaternion coordinates.

5.1. Trajectory results

The results of our design procedure are listed in Table 2. The line $L = \mathbf{B} + t\mathbf{G}$ defines the fixed axis of the optimum design. The average measure of distance between the specified positions and those reached by this design is listed as the ”error.” Cases 1 and 4 list the optimum designs obtained using the dual and double quaternion formulations, respectively.

Figure 4 shows the CC-chain and the trajectory of the end-effector as it followsthe desired trajectory for the dual quaternion formulation. Figure 5 shows the CC-chain and the trajectory of the end-effector resulting from the double quaternion formulationn.

5.2. Effect of change of coordinates

The change of the optimum design with the change in global coordinates is demonstrated in cases 2 and 3 using dual quaternions and cases 5 and 6 using double quaternions. The coordinate transformations con-

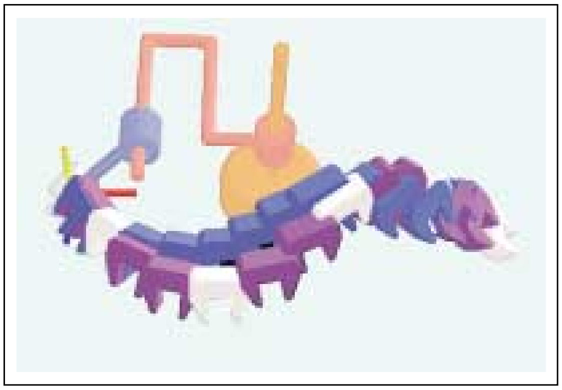

Figure 5: The end-effector trajectory for a CC-chain optimized using double quaternion coordinates.

sists of a translation by $+\mathbf{t} = (8, 8, 8)$ and by $-\mathbf{t} = (-8, -8, -8)$ of the global frame F. The specified trajectory was unchanged.

Using dual quaternions to measure distance error in the design optimization procedure, we obtain different CC chains for different fixed frames shown as the two additional lines in Figure 4. Also compare the coordinates of $\mathbf{G}$ and $\mathbf{B}$ obtained for cases 1, 2, 3 in Table 2. This table also shows that changes of the global frame have a small effect on the distance measured using double quaternions, and we obtain the essentially the same CC-chain for each of the three cases 4, 5, and 6.

6. Summary and Conclusions

This paper presents a design procedure for robots that minimizes the error between a specified taskspace and the workspace reachable by the resulting design. Because the measurement of distance in the group of rigid motions $SE(3)$ is not coordinate frame invariant, we reformulate displacements in $SE(3)$ as rotations in $SO(4)$. This provides a coordinate frame invariant measure of distance for the optimization procedure.

An example demonstrates the use of this procedure to design of a four degree of freedom CC-chain that follows an arbitrarily specified trajectory. We see that the dual quaternion formulation yields different designs for the same trajectory, when the global frame F is changed. The double quaternion formulation yields the same linkage for the three locations of the fixed frame F.

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