DIMENSIONAL SYNTHESIS OF SPATIAL RR ROBOTS

ALBA PEREZ
Robotics and Automation Laboratory
University of California, Irvine
Irvine, CA 92697
email: maperez@uci.edu

AND

J. MICHAEL MCCARTHY
Department of Mechanical and Aerospace Engineering
University of California, Irvine
Irvine, CA 92697
email: jmmccart@uci.edu

Abstract: This paper presents a synthesis procedure for a two-degree-of-freedom spatial RR chain to reach an arbitrary end-effector trajectory. Spatial homogeneous transforms are mapped to 4×4 rotations and interpolated as double quaternions. Each set of three spatial positions obtained from the interpolated task is used to define an RR chain. The RR chain that best fits the trajectory is the desired robot. The procedure yields a unique robot independent of the coordinate frame defined for the task.

1. Introduction

In this paper we present an algorithm to design a spatial RR chain that fits a desired end-effector trajectory. It is known that there is no frame invariant metric for spatial displacements with which to measure this fit, see Park (1995). Therefore, we map the desired trajectory and the workspace of the RR chain to the group of rotations in four dimensional space SO(4) that approximates the group of spatial displacements SE(3) over a specified region of space. McCarthy and Ahlers (1999) demonstrate this design approach for CC chains and show that it is coordinate-frame invariant to a specified accuracy within the task space. This paper combines this approach with a new solution for RR chain synthesis.

Veldkamp (1967) solved the design equations of a spatial RR chain for three instantaneous positions of the end-effector and found that there were two solutions that formed a Bennett linkage. Tsai and Roth (1973) general-

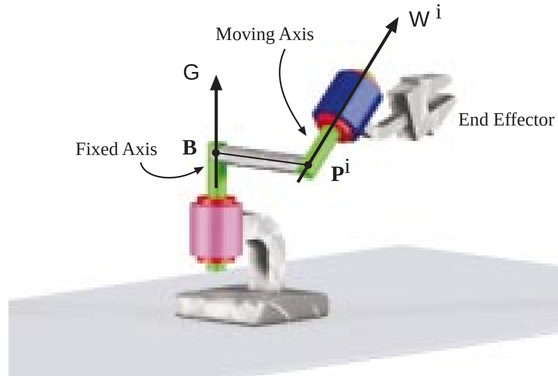

Figure 1. A spatial RR robot.

ized this result to obtain two solutions for three arbitrary spatial positions. This required the solution of ten quadratic equations in ten unknowns. In this paper we introduce the use of the cylindroid associated with a Bennett linkage (Huang 1996) to define a coordinate frame that reduces the design problem to four non-linear equations in four unknowns.

2. The Design Equations for a Spatial RR Chain

A spatial RR robot is a two-degree-of-freedom open chain formed by a fixed revolute joint and a moving revolute joint, Figure 1. The axes of the two joints are skew to each other in space. Let F be the fixed frame and $[G]$ be the 4×4 homogenous transform that locates the base of the RR chain in F. And let M be a moving frame in the end-effector and $[H]$ locate this frame relative to the moving axis. Then the kinematics equation of the RR chain can be formulated following the Denavit-Hartenberg convention to yield

$$[T] = [G][Z(\theta, 0)][X(\alpha, a)][Z(\phi, 0)][H], \tag{1}$$

where $[Z(\theta, 0)]$ is a rotation about the z-axis and $[X(\alpha, a)]$ is a screw-displacement along the x-axis.

Given a set of three task positions for the end-effector, we can compute the relative transformations $[T_{12}] = [A_{12}, \mathbf{d}_{12}]$ and $[T_{13}] = [A_{13}, \mathbf{d}_{13}]$ from the first position to the other two. Let the Plücker coordinates of the axis of the fixed joint be denoted by $\mathsf{G} = (\mathbf{G}, \mathbf{B} \times \mathbf{G})^T = (\mathbf{G}, \mathbf{R})^T$, where $\mathbf{G}$ is the direction of the axis and $\mathbf{R}$ is the moment of the axis relative to the origin of F. The coordinates of the moving axis in the first task position are $\mathsf{W}^1 = (\mathbf{W}^1, \mathbf{P}^1 \times \mathbf{W}^1)^T = (\mathbf{W}^1, \mathbf{V}^1)^T$. The Plücker coordinates of the moving axis in the second and third positions are given by

$$\mathsf{W}^i = [\hat{T}_{1i}]\mathsf{W}^1, \; i = 2, 3. \tag{2}$$

where $[\hat{T}_{1i}]$ is the 6×6 relative transformation matrix for line coordinates.

2.1. THE CONSTRAINT EQUATIONS

The constraint equations for an RR chain are defined by Tsai and Roth (1973) in terms of the equivalent screw triangle. Suh and Radcliffe (1978) present a separate set of design equations based on the geometry of the RR chain. It is possible to show both of these sets of equations are equivalent to the following set of constraint equations.

The angle between the directions $\mathbf{G}$ and $\mathbf{W}^1$ of the fixed and moving axes must be constant as the RR chain moves. Furthermore, the reference point $\mathbf{P}^i$ must lie at a constant distance from the fixed point $\mathbf{B}$. This yields the constraints

$$\mathbf{G} \cdot [A_{1i} - I]\mathbf{W}^1 = 0, \quad (\mathbf{B} - \mathbf{P}^1) \cdot \mathbf{S}_{1i} - \frac{d_{1i}}{2} = 0, i = 2, 3. \tag{3}$$

Note $[T_{1i}] = [A_{1i}, \mathbf{d}_{1i}]$ is 4×4 transform defining the two relative displacements. $\mathbf{S}_{1i}$ is the direction of the relative rotation axis and d_{1i} is the slide along the axis.

In order to ensure that the vector $\mathbf{P}^i - \mathbf{B}$ lies along the normal line between the axes of the two revolute joints, we need the additional six constraints

$$\mathbf{G} \cdot (\mathbf{P}^i - \mathbf{B}) = 0, \quad \mathbf{W}^i \cdot (\mathbf{P}^i - \mathbf{B}) = 0, \ i = 1, 2, 3. \tag{4}$$

The result is the set of ten design equations (3) and (4) for the RR robot.

The design parameters are the four coordinates that define the directions $\mathbf{G}$, $\mathbf{W}^1$ of the fixed and moving axes, and the six coordinates that define the reference points $\mathbf{B}$ and $\mathbf{P}^1$. Thus, we obtain a system of ten quadratic design equations in ten unknowns.

2.2. BENNETT LINKAGE COORDINATES

It is known that the two RR chains obtained from these design equations yield a Bennett linkage (Tsai and Roth 1973). Huang (1996) showed that the relative screw axes of the movement of a Bennett linkage form a cylindroid. Thus, the relative screw axes S_{12} and S_{13} obtained from the three task positions for our RR chain must lie on this cylindroid.

A cylindroid is a ruled surface consisting of lines that intersect an axis at right angles, Figure 2. In each cylindroid it is possible to identify a center point along the axis and principal directions, which form the principal coordinate frame. See Parkin (1997) and Hunt (1978). We reformulate our

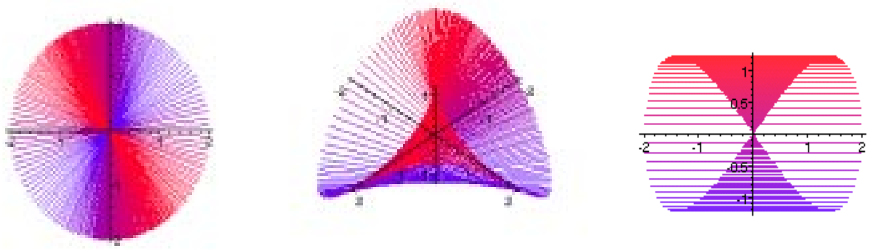

Figure 2. The design cylindroid.

design problem in the principal frame of this cylindroid and use a set of coordinates originally introduced by Yu (1981) in order to simplify the design equations.

Yu identified a tetrahedron formed by a Bennett linkage, see Figure 3, and introduced coordinates closely related to those defined here. Let $\mathbf{B}$, $\mathbf{P}^1$, $\mathbf{Q}$, $\mathbf{C}^1$ be the vertices of this tetrahedron. The axis K of the of the cylindroid is the common normal to the lines defined by $\mathbf{B} - \mathbf{C}^1$ and $\mathbf{P}^1 - \mathbf{Q}$. Let a and b be the parameters

$$a = \frac{|\mathbf{B} - \mathbf{C}^1|}{2}, \quad b = \frac{|\mathbf{P}^1 - \mathbf{Q}|}{2}, \tag{5}$$

and let c and κ be the distance and angle between these edges along K. Then, the principal axis X intersects K at one-half the distance c and is oriented so it bisects that angle κ. The joint axes G and W^1 pass through $\mathbf{B}$ and $\mathbf{P}^1$ and are perpendicular to the faces of the tetrahedron. The axes X, $\mathsf{Y}$and K are the principal axes of the cylindroid.

In this coordinate frame the direction vectors $\mathbf{G}$ and $\mathbf{W}^1$ of the joint axes $\mathsf{G} = (\mathbf{G}, \mathbf{B} \times \mathbf{G})$ and $\mathsf{W}^1 = (\mathbf{W}^1, \mathbf{P}^1 \times \mathbf{W}^1)$ are computed using the condition that they are perpendicular to the respective faces of the tetrahedron, to yield

$$\mathbf{G} = K_g \left\{ \begin{array}{c} 2bc \sin\frac{\kappa}{2} \\ 2bc\cos\frac{\kappa}{2} \\ 4ab\cos\frac{\kappa}{2}\sin\frac{\kappa}{2} \end{array} \right\}, \quad \mathbf{W}^1 = K_w \left\{ \begin{array}{c} -2ac \sin\frac{\kappa}{2} \\ 2ac\cos\frac{\kappa}{2} \\ 4ab\cos\frac{\kappa}{2}\sin\frac{\kappa}{2} \end{array} \right\}, \tag{6}$$

where the constants K_g and K_w normalize the vectors $\mathbf{G}$ and $\mathbf{W}^1$. Thus, the four parameters a, b, c, κ completely define the RR chain. Furthermore, the normal constraint equations (4) are identically satisfied, and the four equations (3) are all we need to design the RR chain.

An algebraic solution of these design equations has been achieved using Maple V. The solution procedure yields two RR chains that combine to form

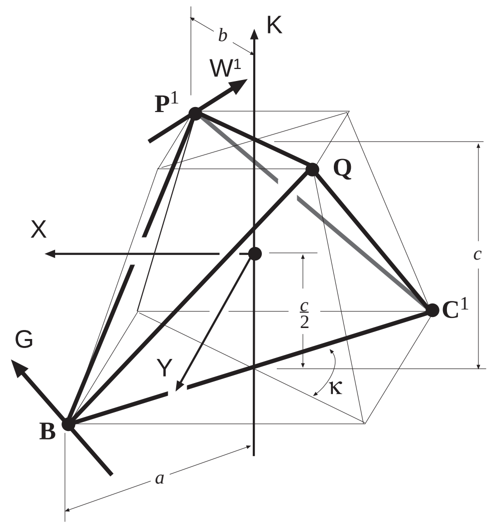

Figure 3. The tetrahedron defines the Bennett linkage.

a Bennett linkage. Please contact the authors to obtain a copy of the Maple worksheet.

3. The Task Specification using Double Quaternions

In order to specify a continuous trajectory for the end-effector of an RR chain, we choose a set of spatial displacements to act as key frames. We then interpolate these key frames using a Bezier formulation based on double quaternions to obtain the complete trajectory.

3.1. HOMOGENEOUS TRANSFORMS AS 4×4 ROTATIONS

A spatial screw displacement of rotation angle θ and translation distance d around and along the z-axis is defined by the 4×4 homogeneous transform,

$$[Z(\theta, d)] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{7}$$

This matrix defines a rigid displacement in the hyperplane $x_4 = 1$ of four dimensional Euclidean space. The same displacement can be defined in a hyperplane $x_4 = R$ parallel by dividing the translation terms by R,

$$[Z(\theta, d)] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & \frac{d}{R} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{8}$$

Now, define the angle γ so that $\tan\gamma = d/R$, then this homogeneous transform can be approximated by the 4×4 rotation

$$[Z(\theta, \gamma)] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\gamma & \sin\gamma \\ 0 & 0 & -\sin\gamma & \cos\gamma \end{bmatrix}. \tag{9}$$

for large R. The approximation is of order $1/R^2$ in the hyperplane of the rigid displacement. Notice that the addition of the $(4, 3)$ element introduces a component that is orthogonal to the displacement.

Associated with each 4×4 rotation is a double quaternion that can be constructed in the same way as the dual quaternion obtained from a 4×4 homogeneous transform. However, in this case the complex unit is denoted by j and has the property that $j^2 = 1$, in contrast to the dual unit ϵ which has the property $\epsilon^2 = 0$. The double quaternion associated with the rotation (9) is given by

$$\tilde{Z}(\theta, \gamma) = \big(0, 0, \sin\frac{\theta + j\gamma}{2}, \cos\frac{\theta + j\gamma}{2}\big)^T. \tag{10}$$

This double quaternion can be reformulated by introducing the symbols $\xi = (1 + j)/2$ and $\eta = (1 - j)/2$, so we have

$$\tilde{Z}(\theta, \gamma) = \xi\big(0, 0, \sin\frac{\theta + \gamma}{2}, \cos\frac{\theta + \gamma}{2}\big)^T + \eta\big(0, 0, \sin\frac{\theta - \gamma}{2}, \cos\frac{\theta - \gamma}{2}\big)^T \tag{11}$$

Because $\xi^2 = \xi$, $\eta^2 = \eta$ and $\xi\eta = 0$, this decomposition allows us to separately manipulate the two quaternions (Clifford 1882).

The set of 4×4 rotations SO(4) has a bi-invariant metric which is now available to measure the distance between our approximations to 4×4 homogeneous transforms. Let L be the maximum dimension of the task space, then the maximum error in this approximation is $\epsilon = (L/R)^2$. Therefore, we can bound this error by choosing R such that

$$R = \frac{L}{\sqrt{\epsilon}}. \tag{12}$$

	x	y	z	θ	ϕ	ψ
M_1	0.0	0.0	0.0	0°	0°	0°
M_2	4.0	2.0	-2.0	5°	0°	15°
M_3	3.0	-2.0	1.0	60°	10°	0°
M_4	8.0	-3.0	2.0	90°	15°	0°

TABLE 1. Key frames for the task trajectory

3.2. THE TASK TRAJECTORY

The task trajectory is defined using key frames which are transformed into approximating 4×4 rotation matrices and their associated double quaternions. We then use Bezier interpolation of each component quaternion separately to generate a continuous trajectory, see Ge and Kang (1995) and Ahlers and McCarthy (2000). The trajectory in three-dimensional space is obtained by converting the 4×4 rotations back to homogeneous transforms.

In this example we specify four key frames to define the end-effector trajectory and interpolate to obtain 22 task positions. The data for the key frames is shown in Table 1 as the location $(x, y, z)^T$ of a reference point and the roll, pitch and yaw orientation angles (θ, ϕ, ψ). Figure 4 shows the interpolated path obtained for this example. The key frames are white and there are six interpolated frames between each key frame.

4. Fitting an RR Robot to the Specified Trajectory

Our design equations provide an analytical solution for an RR robot that fits three positions exactly. Therefore it is easy to compute all RR robots that reach the $\binom{22}{3}$=1540 sets of three positions in the task trajectory. This reduces the design problem to selecting among these chains to find the one that best fits the remaining positions.

4.1. THE DOUBLE KINEMATICS EQUATIONS

We use a double quaternion metric to measure the distance position reachable by the end-effector of an RR robot and the desired frame along the trajectory. This requires that we assemble the double quaternion version of the kinematics equations of the RR robot, given by

$$\tilde{D} = \tilde{G}\tilde{Z}(\theta, 0)\tilde{X}(\alpha, \rho)\tilde{Z}(\phi, 0)\tilde{H}, \qquad (13)$$

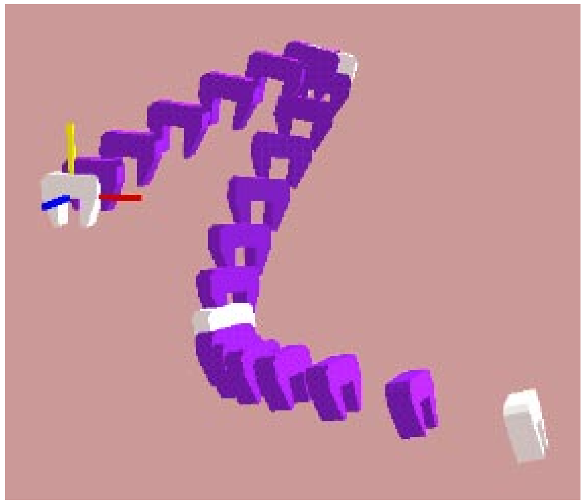

Figure 4. Double quaternion interpolated path

$\mathbf{G} = (0.238, -0.945, 0.224)^T$	$\mathbf{R} = (-5.498, -1.025, 1.528)^T$	
$\mathbf{W}^1 = (0.144, -0.671, -0.727)^T$	$\mathbf{V}^1 = (10.020, -4.849, 6.451)^T$	$\hat{\alpha} = (1.041, 9.795)$

TABLE 2. The fixed axis $\mathsf{G} = (\mathbf{G}, \mathbf{R})^T$ and moving axis $\mathsf{W}^1 = (\mathbf{W}^1, \mathbf{V}^1)^T$ form the desired RR chain.

where $\tilde{Z}$ is given above and $\tilde{X}$ is the double quaternion representing planar rotations in the yz and xw coordinate planes. The double quaternions $\tilde{G}$ and $\tilde{H}$ are known from the location of the fixed and moving axes. The joint parameter values θ and ϕ that locate the closest position $\tilde{D}_i$ of the RR chain to a specific frame $\tilde{P}_i$ of the interpolated path are obtained by minimizing the distance

$$\mathrm{dist}(\tilde{P}_i, \tilde{D}_i) = \sqrt{(\tilde{P}_i - \tilde{D}_i) \cdot (\tilde{P}_i - \tilde{D}_i)}. \tag{14}$$

This metric is invariant to changes in both the fixed and moving coordinate frames (Etzel and McCarthy 1999).

The overall error between the desired trajectory and that attainable by the RR chain is given by $E = \sum_{i=1}^{n} \mathrm{dist}(\tilde{P}_i, \tilde{D}_i)$. The coordinates for the fixed and moving axes of the RR robot are given in Table 2. Figure 5 shows that the end-effector trajectory (light shading) closely follows the desired trajectory (dark shading). The RR robot is shown in Figure 6.

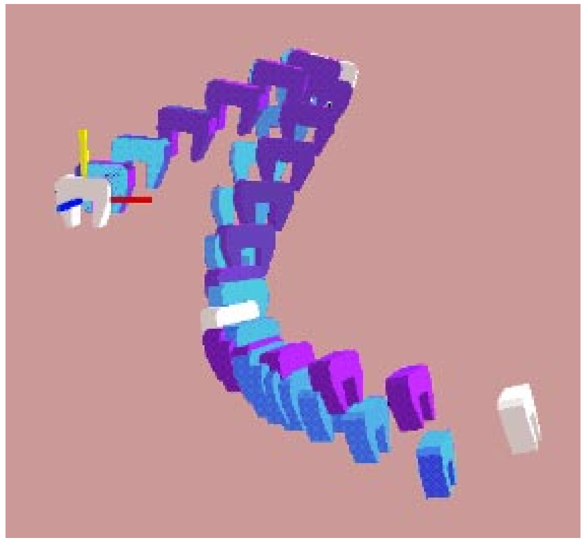

Figure 5. Double quaternion interpolated path and reachable path

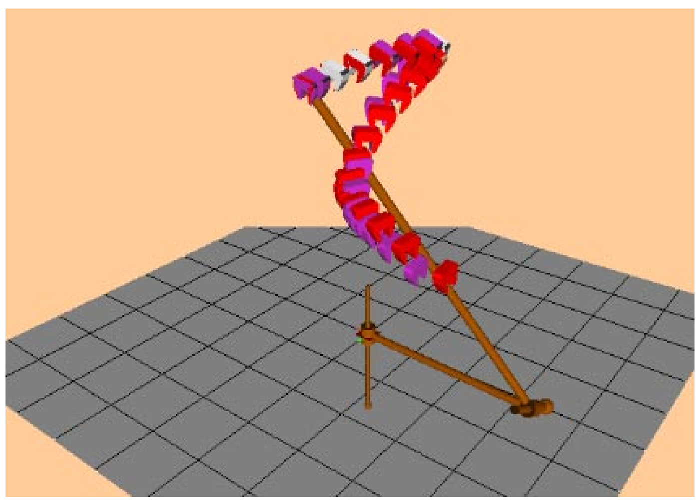

Figure 6. The RR robot designed to fit the task trajectory.

The parameter R provides a weight for the translation component in the distance measurement. For large R the orientation term dominates. As R decreases the influence of the translation increases, however, the error due to changes in coordinate frame also increases. The designer can use this parameter to balance coordinate frame invariance against fitting the translation components of the trajectory.

5. Conclusions

In this paper we design a two-degree-of-freedom spatial RR robot that can best reach a specified spatial trajectory. Double quaternion interpolation is used to obtain a smooth trajectory from a set of key frames. The analytical solution of the design equations provide an RR chain for each set of three positions. The RR chain that minimizes the distance to the remaining positions is our optimum design. The error measurement is done using a double quaternion metric which provides frame independence to a specified accuracy within the task space.

This approach has focused on solving the algebraic constraint equations. However, it may be possible to fit the double kinematics equations directly to the task space. This would provide a framework for the design of spatial 3R robots, as well as general robotic systems.

References

Ahlers, S. G., and McCarthy, J. M., (2000), The Clifford Algebra of Double Quaternions and the Optimization of TS Robot Design *Applications of Clifford Algebras in Computer Science and Engineering*, E. Bayro and G. Sobczyk, editor, (in press).

Clifford, W. K., (1882), Preliminary Sketch of Biquaternions (1873). *Mathematical Papers*, edited by R. Tucker, Macmillan, London, pp. 658.

Etzel, K.R., and McCarthy, J.M., (1999), Interpolation of Spatial Displacements Using the Clifford Algebra of E4 *ASME Journal of Mechanical Design*, vol. 121(1), pp. 39-44.

Ge, Q.J., and Kang, D., (1995), Motion Interpolation with G^2 Composite Bezier Motions. *ASME Journal of Mechanical Design*, vol. 116, pp. 749-755.

Huang, C., (1996), The Cylindroid Associated with Finite Motions of the Bennett Mechanism. *Proc. of the ASME Design Engineering Technical Conferences*, Irvine, CA.

Hunt, K.H.,(1978), *Kinematic Geometry of Mechanisms.* Clarendon Press.

McCarthy, J.M.,(1990), *Introduction to Theoretical Kinematics.* MIT Press.

McCarthy, J.M., and Ahlers, S., (1999), Dimensional Synthesis of Robots Using a Double Quaternion Formulation of the Workspace. *9th. International Symposium of Robotics Research ISRR'99*, pp. 1-6.

Park, F. C, (1995), Distance Metrics on Rigid Body Motions with Applications to Mechanism Design. *ASME J. of Mechanical Design*, vol. 117(1), pp. 48-54.

Parkin, I.A., (1997), Finding the Principal Axes of Screw Systems. *Proc. of the ASME Design Engineering Technical Conferences*, Sacramento, CA.

Shoemake, K.,(1985), Animating Rotation with Quaternion Curves. *ACM Siggraph*, vol. 19(3), pp. 245-254.

Suh, C.H., and Radcliffe, C.W., (1978), *Kinematics and Mechanisms Design.* John Wiley.

Tsai, L.W., and Roth, B.,(1973) A Note on the Design of Revolute-Revolute Cranks. *Mechanisms and Machine Theory*, vol. 8, pp. 23-31.

Veldkamp, G.R., (1967) Canonical Systems and Instantaneous Invariants in Spatial Kinematics. *Journal of Mechanisms*, vol. 3, pp. 329-388.

Yu, H.C., (1981) The Bennett Linkage, its Associated Tetrahedron and the Hyperboloid of its Axes. *Mechanism and Machine Theory*, vol. 16, pp. 105-114.