Then
\[
\gamma = \frac{\frac{1}{2} \rho (dp/\partial x)}{\frac{1}{2} \rho u_{m}^{2}} = \frac{\frac{1}{2} \rho (dp/\partial x)}{\rho (u_{m}^{2} / \nu) (dp/\partial x) u_{m}} = \frac{df}{d(\rho u_{m}^{2})} = \frac{16}{Re}
\]

(5)

(6)

The hyperbola \( \gamma \) versus \( Re \) describes the scale effect of Reynolds number on the pressure drop coefficient. The analysis applies only to laminar flow; at a critical Reynolds number dependent on factors to be taken up in Chapter 17, transition from laminar to turbulent flow occurs and time dependent terms associated with the unsteady character of the flow would have to be included if a rigorous solution were to be obtained.

15.3 Laminar Boundary Layer Along a Flat Plate

The solution of Eq. 14.3 for the steady flow of an incompressible viscous fluid along a flat plate, as shown in Fig. 2, was obtained by Blasius in 1908. For this

\[
\begin{align*}
\frac{\partial u}{\partial y} + \frac{v}{\partial y} &= 0 \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= \frac{1}{2}
\end{align*}
\]

(7)

To solve these equations, we need two boundary conditions for the first and one for the second. The conditions are

\[
\begin{align*}
at \quad y = 0: & \quad u = v = 0 \\
at \quad y = \infty: & \quad u = u_{w}
\end{align*}
\]

(8)

They express the physical conditions that there is no slip at the boundary \( (u = 0 \text{ at } y = 0) \), that the boundary is a streamline \( (v = 0 \text{ at } y = 0) \), and that the horizontal flow is unaffected at infinity \( (u = u_{w} \text{ at } y = \infty) \).

We see that in Eqs. 7 we have two equations to determine the two unknown vari-
ables. In order to get a single unknown variable and a single equation, we introduce the *stream function*, which was defined and discussed in Section 2.6. The stream function \( \psi \) is defined as that function of \( x \) and \( y \) such that

\[
\begin{align*}
    u &= \frac{\partial \psi}{\partial y}, & v &= -\frac{\partial \psi}{\partial x} \\
\end{align*}
\]  

(9)

It is clear that the stream function defined in this way satisfies identically the continuity equation.

We now introduce \( \psi \) as a function of \( x \) and \( y \) such that the equation of motion of Eqs. 7 reduces to an ordinary differential equation, that is, the two independent variables are reduced to one. The reason for seeking an ordinary differential equation is that no general methods exist for solving partial differential equations of the type of the equation of motion. The most distressing feature of the equation is that it is nonlinear in the dependent variable \( \psi \), as is evident if Eqs. 9 are substituted in Eqs. 7.

We therefore seek to express the equation of motion in terms of a single independent variable \( \eta \), a function of \( x \) and \( y \), that is, the equation of motion will be expressed in a form in which neither \( x \) nor \( y \) appears explicitly. Blasius found that if the new variable \( \eta \) were made proportional to \( y/\sqrt{x} \), an ordinary differential equation resulted.*

It is, in general, most convenient to work with dimensionless quantities, and accordingly we define

\[
\eta = \frac{y}{\sqrt{x}}, \quad \psi = \frac{1}{u_{\infty}} \psi \quad (9)
\]

(10)

Here \( \eta \) is dimensionless and \( \psi \) has the dimensions: velocity \( X \) length. We next determine, by means of Eqs. 9 and 10, the terms in Eqs. 7. Differentiations with respect to \( \eta \) are denoted by primes. Then,

\[
\begin{align*}
    u &= \frac{1}{2} u_{\infty} f', & \frac{\partial u}{\partial x} &= \frac{1}{4} \frac{u_{\infty}}{x} \psi f' \\
    \frac{\partial u}{\partial y} &= \frac{1}{4} \left( \frac{u_{\infty}}{x} \right)^{1/2} u_{\infty} f' f'' \\
    \frac{\partial^2 u}{\partial y^2} &= \frac{1}{8} \left( \frac{u_{\infty}}{x} \right)^{1/2} u_{\infty} f'' \\
    v &= \frac{1}{2} \left( \frac{u_{\infty}}{x} \right) (f' - f) \\
    \frac{\partial^2 \psi}{\partial y^2} &= \frac{1}{8} \left( \frac{u_{\infty}}{x} \right)^{1/2} f'' (f' - f)
\end{align*}
\]

(11)

When these values are substituted in the first of Eqs. 7 the result is the differential equation

\[
\begin{align*}
    f'' + f f'' &= 0
\end{align*}
\]

(12)

*The *order-of-magnitude* analysis of Appendix B justifies the choice of \( y/\sqrt{x} \) as the independent variable.

\[
\begin{align*}
    \frac{\partial u}{\partial y} &= \frac{u_{\infty}}{4} \left( \frac{u_{\infty}}{x} \right)^{1/2} f'' \\
    \frac{\partial^2 u}{\partial y^2} &= \frac{u_{\infty}}{8} \left( \frac{u_{\infty}}{x} \right)^{1/2} f''
\end{align*}
\]
and the boundary conditions, Eqs. 8, become

\[ f = 2 \quad \text{at} \quad \eta = \infty \]

and the solution \( f(\eta) \) will, by Eqs. 11, enable the determination of \( u \) and \( v \). The uniqueness of the solution has not been proved, but comparison with experiment has shown that the solution given is the one that describes the flow for the case considered. The differential equation, Eq. 12, appears simple; on the contrary, it is nonlinear and quite difficult. No closed solution has been found and, thus, solution by series is resorted to. We assume a solution of the form

\[ f = A_0 + A_1 \eta + \frac{A_2}{2!} \eta^2 + \frac{A_3}{3!} \eta^3 + \cdots + \frac{A_n}{n!} \eta^n + \cdots \]

When the first two boundary conditions are applied to \( f \), we find that \( A_0 = A_1 = 0 \). After substituting the series for \( f \) into Eq. 12, we get

\[ A_2 + A_4 \eta + \frac{A_6}{2!} \eta^2 + \cdots + \left( \frac{A_2}{2!} \eta^2 + \frac{A_3}{3!} \eta^3 + \cdots \right) \left( A_2 + A_4 \eta + \frac{A_6}{2!} \eta^2 + \cdots \right) = 0 \]

The multiplication is carried out and the coefficients of like powers of \( \eta \) are collected. Then,

\[ A_2 + A_4 \eta + \left( \frac{A_2}{2!} + \frac{A_4}{2!} \right) \eta^2 + \cdots = 0 \]

Since this equation must hold for all values of \( \eta \), the coefficients of every power of \( \eta \) must vanish. Hence,

\[ A_2 = A_4 = 0, \quad A_2^2 + A_4 = 0, \text{etc.} \]

Then all terms can be expressed as functions of \( \eta \) and \( A_2 \):

\[ f = \frac{A_2 \eta^2}{2!} + \frac{A_2^2 \eta^3}{3!} + \frac{11 A_2^3 \eta^4}{4!} + \frac{375 A_2^4 \eta^5}{5!} + \cdots \] (14)

Equation 14 satisfies the first two boundary conditions of Eqs. 13, and the third will be used to determine \( A_2 \).

To accomplish this we write \( f(\eta) \) in the equivalent form

\[
\begin{align*}
    f &= A_2^{1/2} \left[ \left( \frac{A_2^{1/2} \eta^2}{2!} \right) + \frac{11}{81} (A_2^{1/2} \eta)^3 + \frac{375}{111} (A_2^{1/2} \eta)^4 + \cdots \right] \\
    &= A_2^{1/2} g(\Gamma)
\end{align*}
\]

where \( \Gamma = A_2^{1/2} \eta \). The boundary condition (Eqs. 13) to be satisfied at \( \eta = \infty \) is

\[ \lim_{\eta \to \infty} f' = 2 \]
which may be written
\[
\lim_{\Gamma \to \infty} \left[ A_2^{3/2} \frac{g(\Gamma)}{\Gamma'} \right] = 2
\]
where the prime refers to differentiation with respect to \( \Gamma \). But, for \( A_2 > 0 \) when \( \eta \to \infty, \Gamma \to \infty \), and we may write instead of the above
\[
\lim_{\eta \to \infty} \frac{g'(\eta)}{\eta} = \frac{2}{A_2^{3/2}}
\]
or
\[
A_2 = \left[ \frac{2}{\lim_{\eta \to \infty} g'(\eta)} \right]^{2/3}
\]
The right-hand side of this equation is plotted as a function of \( \eta \), and \( A_2 \) can be determined to any desired approximation. Goldstein (1958) found that \( A_2 = 1.32824 \). The quantities \( f, f', \) and \( f'' \) are plotted in Fig. 3 for this value of \( A_2 \).

The solution shows that the value of \( u \) does not reach \( u_e \) until \( \eta = \infty \), that is, at \( y = \infty \). However, at \( \eta = 2.6, u/u_e = 0.994 \); therefore, if we choose the edge of the boundary layer \( (y = \delta) \) as the point where \( u \) is within 1 percent of \( u_e \), we get, from Eqs. 10,
\[
\delta = 5.2 \sqrt{\frac{u_e x}{v}} = \frac{5.2 x}{\sqrt{Re_e}}
\]
(15)
where \( Re_e = u_e x / v \).

\[\text{at } \eta = \infty, \text{ or} \]
\[\text{at edge of Blalux} \]
\[\eta = 2.6, \quad u/u_e = 0.994\]

\[\eta = 2.6, \quad u/u_e = 0.994\]

Fig. 3 The Blalux functions \( f(\eta), f'(\eta), f''(\eta) \).

\[\text{at } \eta = \infty, \text{ or} \]
\[\text{at edge of Blalux} \]
\[\eta = 2.6, \quad u/u_e = 0.994\]
Since the definition of the boundary layer thickness, \( \delta \), is arbitrary, we define a displacement thickness \( \delta^* \) as illustrated for flow along a flat plate in Fig. 4. We see that \( \delta^* \) at \( x = x_1 \) is the amount by which the streamline entering the boundary layer at that point has been displaced outward by the retardation of the flow in the boundary layer. The velocity profile shown at the right illustrates that, since the two cross-hatched areas are equal, the displacement thickness is given by the integral

\[
\delta^* = \int_0^{\infty} \left( 1 - \frac{u}{u_e} \right) dy.
\]

in which \( \delta^* \) is indicated at the surface instead of at the edge of the boundary layer.

We now calculate \( \delta^* \), which, according to Eqs. 11 and 16, is given by

\[
\delta^* = \int_0^{\infty} \left( 1 - \frac{u}{u_e} \right) dy = \left[ \frac{\rho \lambda}{u_e} \right]^{1/2} \int_0^{\infty} (2 - f') d\eta
\]

\[
= \left( \frac{\rho \lambda}{u_e} \right)^{1/2} \left[ 2\eta - \int_0^\infty \left. \frac{d}{d\eta} \left( \frac{\rho \lambda}{u_e} \right)^{1/2} \right|_{\eta=0} \right]
\]

\[
= \left( \frac{\rho \lambda}{u_e} \right)^{1/2} \lim_{\eta \to \infty} \left( 2\eta - \frac{\rho \lambda}{u_e} \right)
\]

Since, from Eqs. 13, \( f'(\infty) = 2 \), the solution for Eq. 12 which must hold for \( \eta \) large is

\( f = 2\eta + \beta \), where \( \beta \) is a constant; that is

\[
\lim_{\eta \to \infty} \left( 2\eta - f' \right) = -\beta
\]

\( \beta \) can be determined from a solution of Eq. 12 by successive approximation. (See Durand, 1943, Vol. 3, p. 87.) The result is \( \beta = -1.7208 \); that is,

\[
\delta^* = \frac{1.7208 x}{\sqrt{Re_x}}
\]

The skin-friction coefficient: \( cf = \frac{\tau_0}{1/2 \rho u^2} \) is calculated as follows:

\[
\tau_0 = \mu \left( \frac{\partial u}{\partial y} \right)_{y=0} = \left( \frac{1}{2} \rho \lambda f''(0) \right) \left( \frac{1}{2} \left( \frac{u_e}{\rho \lambda} \right)^{1/2} \right)
\]

\[
= \frac{1}{4} \kappa A z u_e \left( u_e \over \rho \lambda \right)^{1/2} \rho
\]
Then

\[
\frac{\bar{C}_f}{M} = \frac{A_2}{2} \left( \frac{\nu}{u'_w} \right)^{1/2} = \frac{0.664}{\sqrt{Re}} \tag{18}
\]

The average skin-friction coefficient \( C_f \) for one side of the flat plate of unit width of length \( l \) is given by

\[
C_f = \int_0^l \frac{\tau_0 dx}{\frac{1}{2} \rho u'^2_t l} = \frac{1.328}{\sqrt{Re}} \tag{19}
\]

where \( Re = u'_d/\nu \). Figures 5 and 6 show excellent agreement between theory and experiment for the velocity profile and for the local skin friction coefficient.

---

**Fig. 5** Comparison between theoretical and experimental velocity distributions in the laminar boundary layer on a flat plate. Experiments by Nikuradse (1942) cover Reynolds numbers range 1.00 \( \times \) 10\(^4\) to 7.28 \( \times \) 10\(^6\).

As is well known, the above solution is valid only below a certain Reynolds number, the value of which is dependent on various influences (see Chapter 17). At higher Reynolds numbers, the flow in the boundary layer becomes *turbulent* and the equation of motion describing the flow must strictly include the transient term \( \partial u/\partial t \). The turbulent boundary layer is discussed in Chapter 17. Figure 6 includes a comparison between skin-friction coefficients for laminar and for turbulent flow in the boundary layer.