16.7 Laurent Series

In various applications it is necessary to expand a function \( f(z) \) around points where \( f(z) \) is singular. Taylor's theorem cannot be applied in such cases. A new type of series, known as Laurent series, is necessary. This will be a representation which is valid in an annulus bounded by two concentric circles \( C_1 \) and \( C_2 \) such that \( f(z) \) is analytic in the annulus and at each point of \( C_1 \) and \( C_2 \) (Fig. 329). As in the case of the Taylor series, \( f(z) \) may be singular at some points outside \( C_1 \) and, as the essentially new feature, it may also be singular at some points inside \( C_2 \).

**Laurent's theorem**

If \( f(z) \) is analytic on two concentric circles \( C_1 \) and \( C_2 \) with center \( a \) and in the annulus between them, then \( f(z) \) can be represented by the **Laurent series**

\[
f(z) = \sum_{n=-\infty}^{\infty} b_n(z - a)^n + \sum_{n=1}^{\infty} \frac{c_n}{(z - a)^n}
\]

\[
= b_0 + b_1(z - a) + b_2(z - a)^2 + \ldots + \frac{c_1}{z - a} + \frac{c_2}{(z - a)^2} + \ldots
\]

where

\[
b_n = \frac{1}{2\pi i} \int_C \frac{f(z^*)}{(z^* - a)^{n+1}} \, dz^*, \quad c_n = \frac{1}{2\pi i} \int_C \frac{f(z^*)}{(z^* - a)^n} \, dz^*,
\]

each integral being taken in the counterclockwise sense around any simple closed path \( C \) which lies in the annulus and encircles the inner circle (Fig. 329).

This series converges and represents \( f(z) \) in the open annulus obtained from the given annulus by continuously increasing the circle \( C_1 \) and decreasing \( C_2 \) until each of the two circles reaches a point where \( f(z) \) is singular.

**Fig. 329.** Laurent's theorem

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1. PIERRE ALPHONSE LAURENT (1813–1854), French mathematician.
2. We denote the variable of integration by \( z^* \) because \( z \) is used in \( f(z) \).
Remark. Obviously, instead of (1) and (2) we may write simply

\[(1') \quad f(z) = \sum_{n=0}^{\infty} A_n (z-a)^n\]

where

\[(2') \quad A_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z^*)}{(z^* - a)^{n+1}} dz^*.\]

**Proof of Laurent's theorem.** Let \(z\) be any point in the given annulus. Then from Cauchy's integral formula [cf. (3) in Sec. 14.5] it follows that

\[(3) \quad f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z^*)}{z^* - z} dz^* - \frac{1}{2\pi i} \oint_{C_3} \frac{f(z^*)}{z^* - z} dz^*,\]

where we integrate in the counterclockwise sense. We shall now transform these integrals in a fashion similar to that in Sec. 16.3. Since \(z\) lies inside \(C_1\), the first of these integrals is precisely of the same type as the integral (1), Sec. 16.3. By expanding it and estimating the remainder as in Sec. 16.3 we obtain

\[(4) \quad \frac{1}{2\pi i} \oint_{C_1} \frac{f(z^*)}{z^* - z} dz^* = \sum_{n=0}^{\infty} b_n (z-a)^n\]

where the coefficients are given by the formula

\[(5) \quad b_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z^*)}{(z^* - a)^{n+1}} dz^*\]

and we integrate in the counterclockwise sense. Since \(a\) is not a point of the annulus, the functions \(f(z^*)/(z^* - a)^{n+1}\) are analytic in the annulus. Hence we may integrate along the path \(C\) (cf. in the theorem) instead of \(C_1\), without altering the value of \(b_n\). This proves (2) for all \(n \geq 0\).

In the case of the last integral in (3), the situation is different, since \(z\) lies outside \(C_3\). Instead of (3), Sec. 16.3, we now have

\[(6) \quad \left| \frac{z^* - a}{z - a} \right| < 1,\]

that is, we now have to develop \(1/(z^* - z)\) in powers of \((z^* - a)/(z - a)\) for the resulting series to be convergent. We find

\[
\frac{1}{z^* - z} = \frac{1}{z^* - a} \cdot \frac{1}{(z - a)} = \frac{-1}{(z-a)\left(1 - \frac{z^* - a}{z - a}\right)},
\]

By applying the formula for the sum of a finite geometric progression to the last expression we obtain
\[
\frac{1}{z^*-z} = -\frac{1}{z-a} \left( 1 + \frac{z^*-a}{z-a} + \left( \frac{z^*-a}{z-a} \right)^2 + \cdots + \left( \frac{z^*-a}{z-a} \right)^n \right) \\
- \frac{1}{z-z^*} \left( \frac{z^*-a}{z-a} \right)^{n+1}.
\]

To get the last integral in (3), we multiply this development by \((-1/2\pi i)f(z^*)\) and integrate over \(C_2\). We readily obtain
\[
-\frac{1}{2\pi i} \int_{C_2} \frac{f(z^*)}{z^*-z} \, dz^* = \frac{1}{2\pi i} \left[ \frac{1}{z-a} \int_{C_2} f(z^*) \, dz^* + \int_{C_2} (z^*-a) f(z^*) \, dz^* + \cdots \\
+ \frac{1}{(z-a)^{n+1}} \int_{C_2} (z^*-a)^n f(z^*) \, dz^* \right] + R_n^*(z);
\]

in this representation the last term is of the form
\[
(7) \quad R_n^*(z) = \frac{1}{2\pi i (z-a)^{n+1}} \int_{C_2} (z^*-a)^n f(z^*) \, dz^*.
\]

In the integrals in the braces we may replace the circle \(C_2\) by the aforementioned path \(C\), without altering their values. This establishes Laurent’s theorem provided that
\[
(8) \quad \lim_{n\to\infty} R_n^*(z) = 0.
\]

We prove (8). Since \(z - z^* \neq 0\) and \(f(z)\) is analytic in the annulus and on \(C_2\), the absolute value of the expression \(f(z^*)/(z - z^*)\) in (7) is bounded, say,
\[
\left| \frac{f(z^*)}{z - z^*} \right| < \tilde{M} \quad \text{for all } z^* \text{ on } C_2.
\]

By applying (4) in Sec. 14.2 to (7) and denoting the length of \(C_2\) by \(l\) we thus obtain
\[
|R_n^*(z)| < \frac{1}{2\pi l} \left| \frac{z^*-a}{z-a} \right|^{n+1} \tilde{M} = \frac{\tilde{M} \left| \frac{z^*-a}{z-a} \right|^{n+1}}{2\pi}.
\]

From (6) we see that the expression on the right approaches zero as \(n\) approaches infinity. This proves (8). The representation (1) with coefficients (2) is now established in the given annulus.

Finally let us prove convergence of (1) in the open annulus characterized at the end of the theorem.

We denote the sums of the two series in (1) by \(g(z)\) and \(h(z)\), and the radii of \(C_1\) and \(C_2\) by \(r_1\) and \(r_2\), respectively. Then \(f = g + h\). The first series is a power series. Since it converges in the annulus, it must converge in the entire disk bounded by \(C_1\), and \(g\) is analytic in that disk.

Setting \(\hat{Z} = 1/(z - a)\), the last series becomes a power series in \(\hat{Z}\). The
annulus $r_2 < |z - a| < r_1$, then corresponds to the annulus $1/r_1 < |Z| < 1/r_2$; the new series converges in this annulus and, therefore, in the entire disk $|Z| < 1/r_2$. Since this disk corresponds to $|z - a| > r_2$, the exterior of $C_2$, the given series converges for all $z$ outside $C_2$, and $h$ is analytic for all these $z$. Since $f = g + h$, it follows that $g$ must be singular at all those points outside $C_1$ where $f$ is singular, and $h$ must be singular at all those points inside $C_2$ where $f$ is singular. Consequently, the first series converges for all $z$ inside the circle about $a$ whose radius is equal to the distance of that singularity of $f$ outside $C_1$ which is closest to $a$. Similarly, the second series converges for all $z$ outside the circle about $a$ whose radius is equal to the maximum distance of the singularities of $f$ inside $C_2$. The domain common to both of those domains of convergence is the open annulus characterized at the end of the theorem, and the proof is complete.

It follows that if $f(z)$ is analytic inside $C_2$, the Laurent series reduces to the Taylor series of $f(z)$ with center $a$. In fact, by applying Cauchy's integral theorem to (2) we see that in this case all the coefficients of the negative powers in (1) are zero.

Furthermore, if $z = a$ is the only singular point of $f(z)$ in $C_2$, then the Laurent expansion (1) converges for all $z$ in $C_1$ except at $z = a$. This case occurs frequently and, therefore, is of particular importance. We discuss it later in detail.

The Laurent series of a given analytic function $f(z)$ in its annulus of convergence is unique (cf. Prob. 10, this section). However, $f(z)$ may have different Laurent series in two annuli with the same center (cf. Example 2, below).

The uniqueness is important, because Laurent series usually are not obtained by using (2) for determining the coefficients, but by various other methods. Some of these methods are illustrated by the following examples. If a Laurent series is founded by any such process, it must be the Laurent series of the given function in the given annulus.

**Example 1**

The Laurent series of $z^{3/2}$ with center 0 can be obtained from (2) in Sec. 16.4. Replacing $z$ by $1/z$ in that series, we find

$$z^{3/2} = z^2 \left(1 + \frac{1}{1! z} + \frac{1}{2! z^2} + \ldots\right) = z^2 + z + \frac{1}{2} + \frac{1}{4! z^2} + \frac{1}{47 z^3} + \ldots \quad (|z| > 0).$$

**Example 2. Laurent expansions in different annuli**

Find all Laurent series of the function $f(z) = 1/(1 - z^2)$ with center at $z = 1$. We have $1 - z^2 = -(z - 1)(z + 1)$. Using the geometric series

$$\frac{1}{1 - q} = \sum_{n=0}^{\infty} q^n \quad (|q| < 1),$$

we find

$$\frac{1}{z + 1} = \frac{1}{2 + (z - 1)} = \frac{1}{2} \left[1 - \left(\frac{z - 1}{2}\right)\right]$$

(a)

and

$$\frac{1}{z - 1} = \sum_{n=0}^{\infty} \left(\frac{z - 1}{2}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{(z - 1)^n}{2^n}.$$
this series converges in the disk \(|z - 1|/2| < 1\), that is, \(|z - 1| < 2\). Cf. Fig. 330. Similarly,

\[
\frac{1}{z + 1} = \frac{1}{(z - 1) + 2} = \frac{1}{(z - 1) \left(1 + \frac{2}{z - 1}\right)}
\]

(b)

\[
\frac{1}{z - 1} \sum_{n=1}^{\infty} \left(\frac{2}{z - 1}\right)^n = \sum_{n=1}^{\infty} \frac{(-2)^n}{(z - 1)^{n+1}};
\]

this series converges for \(|2/(z - 1)| < 1\), that is, \(|z - 1| > 2\). Cf. Fig. 330. Hence from (a) we obtain

\[
f(z) = \frac{-1}{(z - 1)(z + 1)} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(z - 1)^{n-1}}{2n+1}
\]

(9)

\[
= \frac{-1/2}{z - 1} + \frac{1}{4} + \frac{1}{8(z - 1)} + \frac{1}{16(z - 1)^3} + \cdots;
\]

this series converges in the domain \(0 < |z - 1| < 2\). Similarly, from (b) we obtain

\[
f(z) = -\sum_{n=0}^{\infty} \frac{(-2)^n}{(z - 1)^{n+2}} = -\frac{1}{(z - 1)^2} + \frac{2}{(z - 1)^3} - \frac{4}{(z - 1)^4} + \cdots + \cdots
\]

This series converges for \(|z - 1| > 2\).

**Example 3. \(\cot z\)**

From Prob. 19 at the end of Sec. 16.4 we immediately have the Laurent series

\[
\cot z = \frac{1}{z} - \frac{1}{3} z - \frac{1}{45} z^3 - \frac{2}{945} z^5 - \cdots \quad (0 < |z| < \pi).
\]

If \(z = a\) is the only singular point of \(f(z)\) in \(C_\theta\) (cf. Fig. 329), the Laurent series (1) converges in a region of the form

\[
0 < |z - a| < R.
\]

The singularity of \(f(z)\) at \(z = a\) is called a pole or an essential singularity depending on whether this Laurent series (the one that converges in a neighborhood of \(z = a\), except at \(z = a\)) has finitely or infinitely many negative powers. An analytic function whose only singularities in the finite plane are poles is called a meromorphic function.

For instance, for determining the kind of singularity of \(1/(1 - z^2)\) at \(z = 1\)
(cf. Example 2) we must use (9) but not (10), since it is (9) that converges in a region of the form (11), with \( a = 1 \). Since (9) has one negative power, that singularity is a pole but not an essential singularity [as we would erroneously conclude from (10)].

A detailed discussion and further examples will follow in the next section.

**Problems for Sec. 16.7**

Expand the following functions in Laurent series which converge for \( 0 < |z| < R \) and determine the precise region of convergence.

1. \( e^{z^2} \)  
2. \( e^{1/z^2} \)  
3. \( \frac{\cos z}{z^2} \)  
4. \( \frac{1}{z^2(1+z)} \)  
5. \( \frac{1}{z^2(1-z^2)} \)  
6. \( \frac{1}{z^2(z-3)} \)  
7. \( \frac{\sin 3z}{z^3} \)  
8. \( \frac{1}{z^3 + z^4} \)  
9. \( \frac{1}{z^2(1+z)^2} \)

10. Prove that the Laurent expansion of a given analytic function in a given annulus is unique.

11. Does \( \tan \left( \frac{1}{z} \right) \) have a Laurent series convergent in a region \( 0 < |z| < R \)?

Find all Taylor series and Laurent series with center \( z = a \) and determine the precise regions of convergence.

12. \( \frac{1}{z^2 + 1} \), \( a = -i \)  
13. \( \frac{1}{z^4} \), \( a = 1 \)  
14. \( \frac{1}{z^3} \), \( a = i \)  
15. \( \frac{1}{z^2 + 1} \), \( a = 1 \)  
16. \( \frac{1}{1 - z^4} \), \( a = -1 \)  
17. \( \frac{4z - 1}{z^2 - 1} \), \( a = 0 \)  
18. \( \frac{\sin z}{(z - 4)^2} \), \( a = \frac{\pi}{4} \)  
19. \( \frac{e^z}{(z - 4)^2} \), \( a = 1 \)  
20. \( \frac{4z^2 + 2z - 4}{z^3 - 4z^2} \), \( a = 2 \)

**16.8 Analyticity at Infinity. Zeros and Singularities**

In this section we consider zeros and singularities of analytic functions. We shall see that there are different types of singularities, which can be characterized by means of the Laurent series.

Our consideration will take place in the extended complex plane since we also want to investigate the behavior of functions \( f(z) \) as \( |z| \to \infty \). So let us first recall from Sec. 13.3 that the extended complex plane is obtained by attaching an improper point \( \infty \) ("point at infinity") to the complex plane. The latter is then called the finite complex plane, for distinction. This process was motivated in Sec. 13.3 by the transformation \( w = 1/z \); then \( z = \infty \) has the image \( w = 0 \) (and \( w = \infty \) has the inverse image \( z = 0 \)).

If we want to investigate a given function \( f(z) \) for large \( |z| \), we may now set \( z = 1/w \) and investigate \( f(1/w) = g(w) \) in a neighborhood of \( w = 0 \). We define

\[
g(0) = \lim_{w \to 0} g(w).
\]