1. Integral Control
Imagine that you use proportional feedback to control the velocity of a DC brushed motor:

\[ \dot{w} + b \, w = \alpha \, v \]

where \( v \) = voltage input to current amplifier that powers the motor, \( w \) = actual angular velocity of motor, sensed with a tachometer, \( \omega_d \) = desired angular velocity, \( K \) = proportional feedback gain, \( \alpha \) = proportionality constant relating \( v \) (i.e. current amplifier input) to torque output from motor, \( b \) = viscous friction.

Draw a block diagram to help you understand the physical parts to the system.

Problem: Show that there is a steady-state error in velocity due to the friction.

Approach 1: Use D.E.
\[
J \, \dot{w} + b \, w = -\alpha K (\omega_d - w)
\]
\[ J \, \dot{w} + (b + K) \, w = \alpha K \, w_d \]

\[
\dot{\omega} \to 0
\]

\[
\omega = \frac{\alpha K \, w_d}{b + \alpha K}
\]

KEY IDEA: We can get rid of this steady-state error by using a proportional plus integral (PI) controller.

\[
\dot{v} = -K_p \, e - K_i \int e \, dt
\]

\[
\dot{\omega} = -\alpha K_p \, e - \alpha K_i \int e \, dt
\]

\[
\dot{e} = \dot{e} \to 0
\]

How does I control work? (try to explain it to your neighbor in words).
Integral control works in the following way:
If error $e(t)$ does not equal zero, then $\int e(t)dt$ increases with time, and eventually the torque (which is proportional to this integral) becomes high enough to overcome friction.

The block diagram for a P-I compensator is:

What is the transfer function for this system?

$$w = \frac{\alpha}{Js + b} \quad v = \frac{\alpha}{Js + b} \left( \frac{k_p s + K_i}{s} \right) (w_d - w)$$

$$\frac{w}{\left( \frac{Js^2 + bs}{Js^2 + bs} + \frac{\alpha k_p s + \alpha K_i}{Js^2 + bs} \right)} = \frac{\alpha k_p s + \alpha K_i}{Js^2 + bs} \frac{w_d}{s}$$

$$C(s) = \frac{\alpha k_p s + \alpha K_i}{Js^2 + bs}$$

This is an example of second order system, which behaves differently than a first order system.

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Important Ideas: integral control can help remove steady state error. However, I-control adds dynamics to the system, which can lead to 2nd order phenomena such as oscillation and resonance.